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A New View of Intra-Regular $\mathcal{AG}$-Groupoids in Terms of Generalized Cubic Ideals

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Abstract

In this paper we characterize the intra-regular $\mathcal{AG}$-groupoids in terms of generalized cubic set. We show that the concept of $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q}\boldsymbol{\Delta})$-cubic ideals and of $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q}\boldsymbol{\Delta})$-cubic interior ideals in an intra-regular $\mathcal{AG}$-groupoid S with left identity coincides. We additionally demonstrate that an $\mathcal{AG}$-groupoid S with left identity is intra-regular if and only if $\beta_1 \wedge^* \beta_2 = \beta_1 \circ^* \beta_2$ for all $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q}\boldsymbol{\Delta})$-cubic quasi ideals $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ of S.

Keywords: $\mathcal{AG}$-groupoids; intra-regular $\mathcal{AG}$-groupoids; cubic sets, generalized sub $\mathcal{AG}$- groupoids; generalized cubic ideals.

1 Introduction

Zadeh [29] started the idea of fuzzy set in 1972, which is a helpful instrument to deal with uncertain, non correct and vague data. Fuzzy set hypothesis is the augmentation of established set hypothesis. Atanassov [2] has given another speculation of fuzzy set as intuitionistic fuzzy sets and also defined different operations [3]. Jun et al. [8] presented another kind of fuzzy sets called cubic sets. The hypothesis of cubic sets pulled in a few mathematicians. Jun et al. [7] considered the hypothesis of cubic sets in different algebraic structures such as cubic subgroups [5], cubic q-ideals of bci-algebras [6] and ideals of bci algebras in cubic structures [9]. Yaqoob et al. [1] examined a few properties of cubic Γ-hyperideals in left almost Γ-semihypergroups and cubic KU-ideals of KU-algebras [27]. For more insight concerning cubic sets and their applications we refer the perusers [12, 13]. Riaz [25] discussed certain properties of bipolar fuzzy soft topology and Malik et al. [14, 15] discussed G-subsets and g-orbits under the action of the modular group. More detail about decision making can be seen in [24, 26]. Murali [16] gave the idea of belongingness of fuzzy point. In [23], the idea of quasi-coincidence of a fuzzy point with a fuzzy set is defined. Recently, Yin and Zhan [28] presented progressively broad types of $(\in, \in \vee q)$-fuzzy filters and define $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filters and gave some intriguing outcomes with regards to terms of these thoughts. The left almost semigroup contracted as a LA-semigroup (also known as Abel-Grassman grouppoids [4]), was first introduced by Kazim and Naseerudin [10]. They summed up some valuable aftereffects of semigroup hypothesis. They presented props on the left of the ternary commutative law $g_1 g_2 g_3 = g_3 g_2 g_1$, to get another pseudo associative law, that is $(g_1 g_2) g_3 = (g_3 g_2) g_1$, and named it as left invertive law. Afterward, Madad et al. [11], Mushtaq and others explored the structure further and added numerous helpful outcomes to the hypothesis of LA-semigroups [21] such as associative LA-semigroups [22], partial ordering and congruences on LA-semigroups [17], On left almost groups [18], m-systems in LA-semigroups [19] and topological structure on LA-semigroups [20].

This article is about the characterizations of Intra-regular $\mathcal{AG}$-groupoids in terms of Generalized Version of Jun's Cubic Sets. We show that the concept of $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic ideals and of $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic interior ideals of an intra-regular $\mathcal{AG}$-groupoid S with left identity coincides. We likewise demonstrate that an $\mathcal{AG}$-groupoid S with left identity is intra-regular if and only if $\beta_1 \wedge^{*} \beta_2 = \beta_1 \circ^{*} \beta_2$ for all $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic quasi ideals $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ of S.

2 Preliminaries

A groupoid $(S, .)$ is called $\mathcal{AG}$-groupoid if its components hold the left invertive law

$$(g_4 g_2) g_3 = (g_3 g_2) g_4.$$

Every $\mathcal{AG}$-groupoid fulfill

$$(g_1 g_2)(g_3 g_4) = (g_1 g_3)(g_2 g_4),$$

for all $g_1, g_2, g_3, g_4 \in S$. If an $\mathcal{AG}$-groupoid contain the left identity, then

$$\begin{aligned} g_1(g_2 g_3) &= g_2(g_1 g_3), \\ (g_1 g_2)(g_3 g_4) &= (g_4 g_2)(g_3 g_1), \\ (g_1 g_2)(g_3 g_4) &= (g_4 g_3)(g_2 g_1). \end{aligned}$$

An $\mathcal{AG}$-groupoid S with left identity is

$$S^2 \subseteq S.$$

An element g_1 of S is called regular if there exist $l_1 \in S$ such that $g_1 = (g_1 l_1)g_1$ and S is called regular, if every element of S is regular. An element g_1 of S is called intra-regular if there exist $l_1, l_2 \in S$ such that $g_1 = (l_1 g_1^2)l_2$ and S is called intra-regular, if every element of S is intra-regular.

Theorem 2.1. *For an $\mathcal{AG}$-groupoid S with left identity, the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *$Q_1 \cap Q_2 = Q_1 Q_2$ for any quasi ideals Q_1 and Q_2 of S.*

Theorem 2.2. *For an $\mathcal{AG}$-groupoid S with left identity, the accompanying conditions are comparable,*

(i) *S is intra-regular.*

(ii) *$R_4 \cap R_5 \subseteq R_4 R_5$ for any left ideal R_4 and right ideal R_5 of S.*

Theorem 2.3. *For an $\mathcal{AG}$-groupoid S with left identity, the accompanying conditions are comparable,*

(i) *S is intra-regular.*

(ii) *$R_3 \cap R_4 = R_4 R_3 (R_3 \cap R_4 \subseteq R_4 R_3)$ for any left ideal R_4 and quasi ideal R_3 of S.*

Theorem 2.4. *For an $\mathcal{AG}$-groupoid S with left identity, the following conditions are equivalent.*

(i) *S is intra-regular.*

(ii) *$R_4 \cap R_3 = (R_4 R_3)R_4$ for any left ideal R_4 and quasi ideal R_3 of S.*

Theorem 2.5. *For an $\mathcal{AG}$-groupoid S with left identity, the accompanying conditions are comparable,*

(i) *S is intra-regular.*

(ii) *$R_2 R_3 \subseteq R_2 \cap R_3$ for all bi ideal R_2 and quasi ideal R_3 of S.*

An interval number is $\widetilde{g_1} = [g_1^-, g_1^+]$, where $0 \leqslant g_1^- \leqslant g_1^+ \leqslant 1$. Let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$, i.e.,

$$D[0,1] = \{\widetilde{g_1} = [g_1^-, g_1^+] : g_1^- \leqslant g_1^+, \text{ for } g_1^-, g_1^+ \in I\}.$$

The operations "$\succeq$ ", " $\preceq$ ", " $=$ ", "$rmin$" and "$rmax$" in case of two elements in $D[0,1]$ defined as. If $\widetilde{g_1} = [g_1^-, g_1^+]$ and $\widetilde{g_2} = [g_2^-, g_2^+] \in D[0,1]$. Then,

(i) $\widetilde{g_1} \succeq \widetilde{g_2}$ if and only if $g_1^- \geq g_2^-$ and $g_1^+ \geq g_2^+$,

(ii) $\widetilde{g_1} \preceq \widetilde{g_2}$ if and only if $g_1^- \leq g_2^-$ and $g_1^+ \leq g_2^+$,

(iii) $\widetilde{g_1} = \widetilde{g_2}$ if and only if $g_1^- = g_2^-$ and $g_1^+ = g_2^+$,

(iv) $rmin\{\widetilde{g_1}, \widetilde{g_2}\} = \left[\min\{g_1^-, g_2^-\}, \min\{g_1^+, g_2^+\}\right]$,

(v) $rmax\{\widetilde{g_1}, \widetilde{g_2}\} = \left[\max\{g_1^-, g_2^-\}, \max\{g_1^+, g_2^+\}\right]$.

An interval valued fuzzy set (briefly, IVF-set) $\widetilde{\hbar}_{R_1}$ on L is defined as

$$\widetilde{\hbar}_{R_1} = \left\{ \left\langle l_1, [\hbar^-_{R_1}(l_1), \hbar^+_{R_1}(l_1)] \right\rangle : l_1 \in L \right\},$$

where $\hbar^-_{R_1}(l_1) \leqslant \hbar^+_{R_1}(l_1)$, for all $l_1 \in L$. Then the ordinary fuzzy sets $\hbar^-_{R_1} : X \to [0,1]$ and $\hbar^+_{R_1} : X \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\widetilde{\hbar}$, respectively. Let $\widetilde{\hbar}_{R_1}(l_1) = [\hbar^-_{R_1}(l_1), \hbar^+_{R_1}(l_1)]$, then,

$$R_1 = \left\{ \left\langle l_1, \widetilde{\hbar}_{R_1}(l_1) \right\rangle : l_1 \in X \right\},$$

where $\widetilde{\hbar}_{R_1} : X \longrightarrow D[0,1]$.

3 $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic ideals

In this segment we have define the idea of a $(\in_\Gamma, \in_\Gamma \vee q\Delta)$-cubic sub $\mathcal{AG}$-groupoid (resp., ideal, bi-ideal, interior ideal and quasi-ideal) of an $\mathcal{AG}$-groupoid which is denoted by S with the assistance of cubic point. Here we give some essential outcomes.

Definition 3.1. [8]. *A cubic set $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ is of the shape:*

$$\beta_1 = \{ \left\langle l_1, \widetilde{\Im}_{\beta_1}(l_1), \hbar_{\beta_1}(l_1) \right\rangle : l_1 \in L \},$$

where the functions $\widetilde{\Im}_{\beta_1} : X \longrightarrow D[0,1]$ and $\hbar_{\beta_1} : X \to [0,1]$.

Definition 3.2. *Let $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ be two cubic sets of S, then $\beta_1 \cap \beta_2 = \left\{ \left\langle l_1, rmin\{\widetilde{\Im}_{\beta_1}(l_1), \widetilde{\Im}_{\beta_2}(l_1)\}, max\{\hbar_{\beta_1}(l_1), \hbar_{\beta_2}(l_1)\} \right\rangle : l_1 \in S \right\}$,*

and $\beta_1 \circ \beta_2 = \left\{ \left\langle l_1, \widetilde{\Im}_{\beta_1 \circ \beta_2}(l_1), \hbar_{\beta_1 \circ \beta_2}(l_1) \right\rangle : l_1 \in S \right\}$,

where

$$\widetilde{\Im}_{\beta_1 \circ \beta_2}(l_1) = \begin{cases} \underset{l_1 = l_2 l_3}{rsup} \{rmin\{\widetilde{\Im}_{\beta_1}(l_2), \widetilde{\Im}_{\beta_2}(l_3)\}\} & \text{if } l_1 = l_2 l_3 \\ [0,0] & \text{otherwise} \end{cases}$$

$$\hbar_{\beta_1 \circ \beta_2}(l_1) = \begin{cases} \inf_{l_1 = l_2 l_3} \{\max\{\hbar_{\beta_1}(l_2), \hbar_{\beta_2}(l_3)\}\} & \text{if } l_1 = l_2 l_3 \\ 1 & \text{otherwise.} \end{cases}$$

Definition 3.3. [12]. *Let $\widetilde{\alpha} \in D(0,1]$ and $\beta \in [0,1)$ such that $\widetilde{0} \prec \widetilde{\alpha}$ and $\beta < 1$, then by cubic point (CP) we mean $l_{1(\widetilde{\alpha},\beta)}(l_2) = \langle l_{1\widetilde{\alpha}}(l_2), l_{1\beta}(l_2) \rangle$ where*

$$l_{1\widetilde{\alpha}}(l_2) = \begin{cases} \widetilde{\alpha} & \text{if } l_1 = l_2 \\ \widetilde{0} & \text{otherwise} \end{cases} \quad \text{and} \quad l_{1\beta}(l_2) = \begin{cases} 0 & \text{if } l_1 = l_2 \\ 1 & \text{otherwise.} \end{cases}$$

For any cubic set $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and for a cubic point $l_{1(\widetilde{\alpha},\beta)}$, with the condition that $[\alpha, \beta] + [\alpha, \beta] = [2\alpha, 2\beta]$ such that $2\beta \leq 1$, we mean

(*i*) $l_{1(\widetilde{\alpha},\beta)} \in_\Gamma \beta_1$ if $\widetilde{\Im}_{\beta_1}(l_1) \succeq \widetilde{\alpha} \succ \widetilde{\gamma_1}$ and $\hbar_{\beta_1}(l_1) \leq \beta < \gamma_2$.

(*ii*) $l_{1(\widetilde{\alpha},\beta)} q\Delta\beta_1$ if $\widetilde{\Im}_{\beta_1}(l_1) + \widetilde{\alpha} \succ 2\widetilde{\delta_1}$ and $\hbar_{\beta_1}(l_1) + \beta < 2\delta_2$.

(*iii*) $l_{1(\widetilde{\alpha},\beta)} \in_\Gamma \vee q\Delta\beta_1$ if $l_{1(\widetilde{\alpha},\beta)} \in_\Gamma \beta_1$ or $l_{1(\widetilde{\alpha},\beta)} q\Delta\beta_1$.

Definition 3.4. [12]. *Let S be an $\mathcal{AG}$-groupoid. Then the cubic characteristic function*

$$\varkappa_\Gamma^\Delta \beta_1 = \left\langle \widetilde{\Im}_{\varkappa_\Gamma^\Delta \beta_1}, \hbar_{\varkappa_\Gamma^\Delta \beta_1} \right\rangle$$

of $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is defined as*

$$\widetilde{\Im}_{\varkappa_\Gamma^\Delta \beta_1}(l_1) \succeq \begin{cases} \widetilde{\delta_1} = [1,1] & \text{if } l_1 \in \beta_1 \\ \widetilde{\gamma_1} = [0,0] & \text{if } l_1 \notin \beta_1 \end{cases} \quad \text{and} \quad \hbar_{\varkappa_\Gamma^\Delta \beta_1}(l_1) \leq \begin{cases} \delta_2 = 0 & \text{if } l_1 \in \beta_1 \\ \gamma_2 = 1 & \text{if } l_1 \notin \beta_1 \end{cases}$$

where $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} \prec \widetilde{\delta_1}$, *and* $\delta_2, \gamma_2 \in [0,1]$ *such that* $\delta_2 < \gamma_2$.

We define a relation $\sqsubseteq \vee_{q(\Gamma,\Delta)}$ on two cubic sets $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ in this way, i.e $\beta_1 \sqsubseteq \vee_{q(\Gamma,\Delta)} \beta_2$ if $l_{1(\widetilde{\alpha},\beta)} \in_\Gamma \beta_1$ implies that $l_{1(\widetilde{\alpha},\beta)} \in_\Gamma \vee q\Delta\beta_2$, $\forall\, l_1 \in S$.

Lemma 3.1. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *be two cubic sets then* $\beta_1 \sqsubseteq \vee_{q(\Gamma,\Delta)} \beta_2$ *if and only if*

$$\begin{aligned} \text{rmax}\left\{\widetilde{\Im}_{\beta_2}(g_1), \widetilde{\gamma_1}\right\} &\succeq \text{rmin}\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{\delta_1}\} \text{ and} \\ \min\{\hbar_{\beta_2}(g_1), \gamma_2\} &\leq \max\{\hbar_{\beta_1}(g_1), \delta_2\}, \end{aligned}$$

where $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} \prec \widetilde{\delta_1}$ *and* $\delta_2, \gamma_2 \in [0,1]$ *such that* $\delta_2 < \gamma_2$.

Proof. Same as in [12]. □

Corollary 3.1. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$, $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *and* $\Omega = \left\langle \widetilde{\Im}_{\Omega}, \hbar_{\Omega} \right\rangle$ *be cubic sets such that*

$$\beta_1 \sqsubseteq \vee_{q(\Gamma,\Delta)} \beta_2 \;\; \text{and} \;\; \beta_2 \sqsubseteq \vee_{q(\Gamma,\Delta)} \Omega,$$

then $\beta_1 \sqsubseteq \vee_{q(\Gamma,\Delta)} \Omega$.

Proof. It follows from the Lemma 3.1. □

Remark 3.1. *The relation $=_{(\Gamma,\Delta)}$ is an equivalence relation on S. Two cubic sets $\beta_2 =_{(\Gamma,\Delta)} \Omega$ if and only if*

$$rmax\{rmin\{\widetilde{\Im}_{\beta_2}(l_1), \widetilde{\delta_1}\}, \widetilde{\gamma_1}\} = rmax\{rmin\{\widetilde{\Im}_{\Omega}(l_1), \widetilde{\delta_1}\}, \widetilde{\gamma_1}\},$$

and

$$\min\{\max\{\hbar_{\beta_2}(l_1), \delta_2\}, \gamma_2\} = \min\{\max\{\hbar_{\Omega}(l_1), \delta_2\}, \gamma_2\}.$$

Definition 3.5. *A cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *of S is said to be* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub $\mathcal{AG}$-groupoid of S if*

$$l_{1(\widetilde{t_1},s_1)} \in_\Gamma \beta_1 \text{ and } l_{2(\widetilde{t_2},s_2)} \in_{(\widetilde{\gamma_2},\gamma_2)} \beta_1,$$

implies that

$$(l_1 l_2)_{\langle rmin\{\widetilde{t_1},\widetilde{t_2}\},\ \max\{s_1,s_2\}\rangle} \in_\Gamma \vee q\Delta\beta_1,$$

where $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} < \widetilde{\delta_1}$, *and* $\delta_2, \gamma_2 \in [0,1]$ *such that* $\delta_2 < \gamma_2$.

Theorem 3.1. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be the cubic set in* S *then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is said to be* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub* $\mathcal{AG}$*-groupoid of* S *if and only if*

$$\text{rmax}\left\{\widetilde{\Im}_{\beta_1}(l_1 l_2), \widetilde{\gamma_1}\right\} \succeq \text{rmin}\{\widetilde{\Im}_{\beta_1}(l_1), \widetilde{\Im}_{\beta_1}(l_2), \widetilde{\delta_1}\} \text{ and}$$
$$\min\{\hbar_{\beta_1}(l_1 l_2), \gamma_2\} \leq \max\{\hbar_{\beta_1}(l_1), \hbar_{\beta_1}(l_2), \delta_2\},$$

where $\widetilde{\delta_1}$ *and* $\widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} \prec \widetilde{\delta_1}$*, and* $\delta_2, \gamma_2 \in [0,1)$ *such that* $\delta_2 < \gamma_2$.

Proof. Similar to the proof of Lemma 3.1. □

Example 3.1. *Let* $S = \{1,2,3\}$ *and the binary operation "·" be defined on* S *as follows:*

$\cdot$	1	2	3
1	2	2	2
2	3	3	3
3	3	3	3

Then $(S, \cdot)$ *is an* $\mathcal{AG}$*-groupoid with no left identity. Define a cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *as follows:*

S	$\widetilde{\Im}_{\beta_1}$	$\hbar_{\beta_1}$
1	$[0.2, 0.3]$	0.6
2	$[0.4, 0.5]$	0.5
3	$[0.6, 0.7]$	0.4

Let us define

$\widetilde{\delta_1} = [0.81, 0.0.85]$	$\gamma_2 = 0.3$
$\widetilde{\gamma_1} = [0.75, 0.8]$	$\delta_2 = 0.2$

such that $\widetilde{\gamma_1} \prec \widetilde{\delta_1}$ *and* $\delta_2 < \gamma_1$*. Then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_{([0.75,0.8],0.3)}, \in_{([0.75,0.8],0.3)} \vee q_{([0.81,0.0.85],0.2)})$*-cubic sub* $\mathcal{AG}$*-groupoid of* S.

Definition 3.6. *A cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *of* S *is said to be* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic left (resp., right) ideal of* S *if* $l_{1(\widetilde{t},s)} \in_\Gamma \beta_1$*, and* $l_2 \in S$ *implies that* $(l_2 l_1)_{\langle \widetilde{t},\, s\rangle} \in_\Gamma \vee q\Delta\beta_1$ *(resp.,* $(l_1 l_2)_{\langle \widetilde{t},\, s\rangle} \in_\Gamma \vee q\Delta\beta_1$*), where* $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} < \widetilde{\delta_1}$ *and* $\delta_2, \gamma_1 \in [0,1]$ *such that* $\delta_2 < \gamma_1$.

$\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ is said to be an $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic ideal if it is both $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic left and $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic right ideal of S.

Definition 3.7. *A cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *of* S *is said to be* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic generalized bi-ideal of* S *if for all* $l_1, l_2, l_3 \in S$ *and* $\widetilde{t_1}, \widetilde{t_2} \in D(0,1]$ *and* $s_1, s_2 \in [0,1]$ *we have,*

$$l_{1(\widetilde{t_1},s_1)} \in_\Gamma \beta_1, l_{3(\widetilde{t_2},s_2)} \in_{(\widetilde{\gamma_2},\gamma_2)} \beta_1 \text{ implies that } ((l_1 l_2) l_3)_{\langle \min\{\widetilde{t_1},\widetilde{t_2}\},\ \max\{s_1,s_2\}\rangle} \in_\Gamma \vee q\Delta\beta_1.$$

Definition 3.8. *A cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *of* S *is said to be* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic bi-ideal of* S *if for all* $l_1, l_2, l_3 \in S$ *and* $\widetilde{t_1}, \widetilde{t_2} \in D(0,1]$ *and* $s_1, s_2 \in [0,1]$ *we have,*

(i) $l_{1(\widetilde{t_1},s_1)} \in_\Gamma \beta_1, l_{2(\widetilde{t_2},s_2)} \in_{(\widetilde{\gamma_2},\gamma_2)} \beta_1$ *implies that* $(l_1 l_2)_{\langle \min\{\widetilde{t_1},\widetilde{t_2}\},\ \max\{s_1,s_2\}\rangle} \in_\Gamma \vee q\Delta\beta_1$

(ii) $l_{1(\widetilde{t_1},s_1)} \in_\Gamma \beta_1, l_{3(\widetilde{t_2},s_2)} \in_{(\widetilde{\gamma_2},\gamma_2)} \beta_1$ *implies that* $((l_1 l_2) l_3)_{\langle \min\{\widetilde{t_1},\widetilde{t_2}\},\ \max\{s_1,s_2\}\rangle} \in_\Gamma \vee q\Delta\beta_1$.

Definition 3.9. *A cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *of* S *is said to be* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic interior-ideal of* S *if for all* $l_1, l_2, l_3 \in S$ *and* $\widetilde{t_1}, \widetilde{t_2} \in D(0,1]$ *and* $s_1, s_2 \in [0,1]$ *we have,*

(i) $l_{1(\widetilde{t_1}, s_1)} \in_\Gamma \beta_1, l_{2(\widetilde{t_2}, s_2)} \in_{(\widetilde{\gamma_2}, \gamma_2)} \beta_1$ *implies that* $(l_1 l_2)_{\langle \min\{\widetilde{t_1}, \widetilde{t_2}\},\ \max\{s_1, s_2\}\rangle} \in_\Gamma \vee q\Delta\beta_1$

(ii) $l_{2(\widetilde{t_1}, s_1)} \in_\Gamma \beta_1$ *implies that* $((l_1 l_2) l_3)_{\langle \min\{\widetilde{t_1}, \widetilde{t_2}\},\ \max\{s_1, s_2\}\rangle} \in_\Gamma \vee q\Delta\beta_1$,

where $\widetilde{\delta_1}$ *and* $\widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} < \widetilde{\delta_1}$, *and* $\delta_2, \gamma_2 \in [0,1]$ *such that* $\delta_2 < \gamma_2$.

Definition 3.10. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be the cubic set in* S *then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is said to be* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic quasi-ideal of* S *if it satisfies* $rmax\left\{\widetilde{\Im}_{\beta_1}(l_1), \widetilde{\gamma_1}\right\} \succeq rmin\{\left(\widetilde{\Im}_{\beta_1} \circ \widetilde{\Im}_{\mathcal{S}}\right)(l_1), \left(\widetilde{\Im}_{\mathcal{S}} \circ \widetilde{\Im}_{\beta_1}\right)(l_1), \widetilde{\delta_1}\}$

and $\min\{\hbar_{\beta_1}(l_1), \gamma_2\} \leq \max\{(\hbar_{\beta_1} \circ \hbar_{\mathcal{S}})(l_1), (\hbar_{\mathcal{S}} \circ \hbar_{\beta_1})(l_1), \delta_2\}$,

where £$= \left\langle \widetilde{\Im}_{\mathcal{S}}, \hbar_{\mathcal{S}} \right\rangle = \langle \widetilde{1}, 0 \rangle$, $\widetilde{\delta_1}$ *and* $\widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} < \widetilde{\delta_1}$, *and* $\delta_2, \gamma_2 \in [0,1]$ *such that* $\delta_2 < \gamma_2$.

Theorem 3.2. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be the cubic set in* S *then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is said to be* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic ideal of* S *if and only if*

$$rmax\left\{\widetilde{\Im}_{\beta_1}(l_1 l_2), \widetilde{\gamma_1}\right\} \succeq rmin\{rmax\left\{\widetilde{\Im}_{\beta_1}(l_1), \widetilde{\Im}_{\beta_1}(l_2)\right\}, \widetilde{\delta_1}\},$$

and

$$\min\{\hbar_{\beta_2}(l_1 l_2), \gamma_2\} \leq \max\{\min\{\hbar_{\beta_1}(l_1), \hbar_{\beta_1}(l_2)\}, \delta_2\},$$

where $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} < \widetilde{\delta_1}$, *and* $\delta_2, \gamma_2 \in [0,1)$ *such that* $\delta_2 < \gamma_2$.

Proof. Same as in [11]. □

Example 3.2. *If we consider the* $\mathcal{AG}$*-groupoid as in Example 3.1 and define the cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *as follows:*

S	$\widetilde{\Im}_{\beta_1}$	$\hbar_{\beta_1}$
1	$[0.2, 0.3]$	0.5
2	$[0.6, 0.7]$	0.3
3	$[0.6, 0.7]$	0.3

with

$\widetilde{\delta_1} = [0.45, 0.5]$	$\gamma_2 = 0.45$
$\widetilde{\gamma_1} = [0.3, 0.4]$	$\delta_2 = 0.4$

such that $\widetilde{\gamma_1} \prec \widetilde{\delta_1}$ *and* $\delta_2 < \gamma_1$. *Then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic ideal of* S.

Lemma 3.2. *Every* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic ideal of* S *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic sub* $\mathcal{AG}$*-groupoid of* S *but converse is not true as shown in the following example.*

Example 3.3. *Let* $S = \{1, 2, 3\}$ *and the binary operation "*$\cdot$*" be defined on* S *as follows:*

$\cdot$	1	2	3
1	3	1	2
2	2	3	1
3	1	2	3

Then $(S,\cdot)$ *is an* $\mathcal{AG}$*-groupoid with* 3 *as a left identity. Define a cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *as follows:*

S	$\widetilde{\Im}_{\beta_1}$	$\hbar_{\beta_1}$
1	$[0.3, 0.4]$	0.6
2	$[0.3, 0.4]$	0.5
3	$[0.5, 0.6]$	0.4

Let us define

$\widetilde{\delta_1} = [0.6, 0.7]$	$\gamma_2 = 0.3$
$\widetilde{\gamma_1} = [0.2, 0.3]$	$\delta_2 = 0.2$

such that $\widetilde{\gamma_1} \prec \widetilde{\delta_1}$ *and* $\delta_2 < \gamma_2$*. Then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_{([0.2,0.3],0.3)}, \in_{([0.2,0.3],0.3)} \vee q_{([0.6,0.7],0.2)})$*-cubic sub* $\mathcal{AG}$*-groupoid of* S*. But* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is not an* $(\in_{([0.2,0.3],0.3)}, \in_{([0.2,0.3],0.3)} \vee q_{([0.6,0.7],0.2)})$*-cubic ideal of* S*. This is due to*

$$\begin{aligned} rmax\left\{\widetilde{\Im}_{\beta_1}(2\cdot 3), \widetilde{\gamma_1}\right\} &= rmax\left\{\widetilde{\Im}_{\beta_1}(1), \widetilde{\gamma_1}\right\} \\ &= rmax\{[0.3,0.4], [0.2,0.3]\} = [0.3,0.4] \\ &\not\succeq rmin\{rmax\left\{\widetilde{\Im}_{\beta_1}(2), \widetilde{\Im}_{\beta_1}(3)\right\}, \widetilde{\delta_1}\} \\ &= rmin\{rmax\{[0.3,0.4],[0.5,0.6]\},[0.6,0.7]\} \\ &= rmin\{[0.5,0.6],[0.6,0.7]\} = [0.5,0.6]. \end{aligned}$$

Corollary 3.2. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be the cubic set in* S *then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is said to be an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic left ideal of* S *if and only if*

$$\begin{aligned} rmax\left\{\widetilde{\Im}_{\beta_1}(l_1 l_2), \widetilde{\gamma_1}\right\} &\succeq rmin\{\widetilde{\Im}_{\beta_1}(l_1), \widetilde{\delta_1}\} and \\ \min\{\hbar_{\beta_1}(l_1 l_2), \gamma_2\} &\leq \max\{\hbar_{\beta_1}(l_1),\ \delta_2\}, \end{aligned}$$

where $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} < \widetilde{\delta_1}$*, and* $\delta_2, \gamma_2 \in [0,1)$ *such that* $\delta_2 < \gamma_2$*.*

Proof. The proof is straightforward. □

Corollary 3.3. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be the cubic set in* S *then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is said to be* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic right ideal of* S *if and only if*

$$\begin{aligned} rmax\left\{\widetilde{\Im}_{\beta_1}(l_1 l_2), \widetilde{\gamma_1}\right\} &\succeq rmin\{\widetilde{\Im}_{\beta_1}(l_1), \widetilde{\delta_1}\} and \\ \min\{\hbar_{\beta_1}(l_1 l_2), \gamma_2\} &\leq \max\{\hbar_{\beta_1}(l_1),\ \delta_2\}, \end{aligned}$$

where $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} < \widetilde{\delta_1}$*, and* $\delta_2, \gamma_2 \in [0,1)$ *such that* $\delta_2 < \gamma_2$*.*

Proof. The proof is straightforward. □

Theorem 3.3. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic sub* $\mathcal{AG}$*-groupoid of* S*. Then the set* $S_{\langle \widetilde{0},1\rangle} = \left\{ l_1 \in S |\ \widetilde{\Im}_{\beta_1}(l_1) \succeq \widetilde{t_1} > \widetilde{0} \text{ and } \hbar_{\beta_1}(l_1) \leq t_1 < 1 \right\}$ *is a sub* $\mathcal{AG}$*-groupoid of* S*.*

Proof. Same as in [11]. □

Theorem 3.4. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *of an* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic left (resp., right) ideal of* S*. Then the set* $S_{\langle \widetilde{0},1 \rangle} = \{l_1 \in S |\ \widetilde{\Im}_{\beta_1}(l_1) \succeq \widetilde{t_1} > \widetilde{0}$ *and* $\hbar_{\beta_1}(l_1) \leq t_1 < 1\}$ *cubic left (resp., right) ideal of* S*.*

Proof. The confirmation is direct. □

Theorem 3.5. *Let* R_1 *be a sub* $\mathcal{AG}$*-groupoid (resp., ideal, bi-ideal, interior ideal and quasi-ideal) of* S*, and let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be the cubic set in* S*. If*

$$\begin{aligned} \widetilde{\Im}_{\beta_1}(l_1) &\succeq \widetilde{0.5} \text{ and } \hbar_{\beta_1}(l_1) \leq 0.5, \text{ if } l_1 \in R_1, \\ \widetilde{\Im}_{\beta_1}(l_1) &= \widetilde{0} \text{ and } \hbar_{\beta_1}(l_1) = 1, \text{ if } l_1 \notin R_1, \end{aligned}$$

then $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub* $\mathcal{AG}$*-groupoid (resp., ideal, bi-ideal, interior ideal and quasi-ideal) of* S*.*

Proof. Same as in [11]. □

Lemma 3.3. *Let* $\emptyset \neq R_1 \subseteq S$*, then* R_1 *is a sub* $\mathcal{AG}$*-groupoid of* S *if and only if cubic characteristic function* $\varkappa_\Gamma^\Delta R_1 = \left\langle \widetilde{\Im}_{\varkappa_\Gamma^\Delta R_1}, \hbar_{\varkappa_\Gamma^\Delta R_1} \right\rangle$ *of* $R_1 = \left\langle \widetilde{\Im}_{R_1}, \hbar_{R_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub* $\mathcal{AG}$*-groupoid of* S*. Where* $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} < \widetilde{\delta_1}$*, and* $\delta_2, \gamma_2 \in [0,1]$ *such that* $\delta_2 < \gamma_2$*.*

Proof. Same as in [11]. □

Theorem 3.6. *The intersection of any two* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub* $\mathcal{AG}$*-groupoids (resp., ideals, bi-ideals, interior ideals and quasi-ideals) of* S *is an* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub* $\mathcal{AG}$*-groupoid (resp., ideal, bi-ideal, interior ideal and quasi-ideal) of* S*.*

Proof. Straightforward. □

Remark 3.2. *The intersection of any family of* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub* $\mathcal{AG}$*-groupoids (resp., ideal) of* S *is an* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub* $\mathcal{AG}$*-groupoid (resp., ideal) of* S*.*

Let us now define the $\in_\Gamma \vee q\Delta$-cubic level set for the cubic set $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ as

$$[\beta_1]_{(\widetilde{t},\delta)} = \{l_1 \in S : l_{1(\widetilde{t},\delta)} \in_\Gamma \vee q\Delta\beta_1\}.$$

Theorem 3.7. *A cubic set* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is said to be* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic sub* $\mathcal{AG}$*-groupoid (resp., ideal, bi-ideal, interior ideal and quasi-ideal) of* S *if and only if* $\emptyset \neq [\beta_1]_{(\widetilde{t},\delta)}$ *is a sub* $\mathcal{AG}$*-groupoid (resp., ideal, bi-ideal, interior ideal and quasi-ideal) of* S*.*

Proof. Same as in [11]. □

Theorem 3.8. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be the cubic set in* S *then* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is said to be* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic bi-ideal of* S *if and only if*

$$\begin{aligned} rmax\left\{\widetilde{\Im}_{\beta_1}(l_1 l_2), \widetilde{\gamma_1}\right\} &\succeq rmin\{rmax\left\{\widetilde{\Im}_{\beta_1}(l_1), \widetilde{\Im}_{\beta_1}(l_2)\right\}, \widetilde{\delta_1}\} \\ \min\{\hbar_{\beta_1}(l_1 l_2), \gamma_2\} &\leq \max\{\min\{\hbar_{\beta_1}(l_1), \hbar_{\beta_1}(l_2)\}, \delta_2\}, \end{aligned}$$

and

$$\begin{aligned} rmax\left\{\widetilde{\Im}_{\beta_1}(l_1l_2l_3),\widetilde{\gamma_1}\right\} &\succeq rmin\{rmax\left\{\widetilde{\Im}_{\beta_1}(l_1),\widetilde{\Im}_{\beta_1}(l_3)\right\},\widetilde{\delta_1}\} \\ \min\{\hbar_{\beta_1}(l_1l_2l_3),\gamma_2\} &\leq \max\{\min\{\hbar_{\beta_1}(l_1),\hbar_{\beta_1}(l_3)\},\delta_2\}, \end{aligned}$$

where $\widetilde{\delta_1},\widetilde{\gamma_1}\in D(0,1]$ *such that* $\widetilde{\gamma_1}<\widetilde{\delta_1}$, *and* $\delta_2,\gamma_2\in[0,1)$ *such that* $\delta_2<\gamma_2$.

Proof. It follows from the proof of Theorem 3.2. □

Corollary 3.4. *A cubic set* $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ *of a* $\mathcal{AG}$*-groupoid* S *is said to be* $(\in_\Gamma,\in_\Gamma\vee \boldsymbol{q}\boldsymbol{\Delta})$*-cubic generalized bi-ideal of* S *if and only if*

$$rmax\left\{\widetilde{\Im}_{\beta_1}(l_1l_2l_3),\widetilde{\gamma_1}\right\}\succeq rmin\{rmax\left\{\widetilde{\Im}_{\beta_1}(l_1),\widetilde{\Im}_{\beta_1}(l_3)\right\},\widetilde{\delta_1}\},$$

and

$$\min\{\hbar_{\beta_1}(l_1l_2l_3),\gamma_2\}\leq\max\{\min\{\hbar_{\beta_1}(l_1),\hbar_{\beta_1}(l_3)\},\delta_2\},$$

where $\widetilde{\delta_1}$ *and* $\widetilde{\gamma_1}\in D(0,1]$ *such that* $\widetilde{\gamma_1}<\widetilde{\delta_1}$, *and* $\delta_2,\gamma_2\in[0,1]$ *such that* $\delta_2<\gamma_2$.

Proof. The confirmation is direct. □

Theorem 3.9. *Let* $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ *be the cubic set in* S *then* $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ *is said to be* $(\in_\Gamma,\in_\Gamma\vee \boldsymbol{q}\boldsymbol{\Delta})$*-cubic interior-ideal of* S *if and only if*

$$\begin{aligned} rmax\left\{\widetilde{\Im}_{\beta_1}(l_1l_2),\widetilde{\gamma_1}\right\} &\succeq rmin\{rmax\left\{\widetilde{\Im}_{\beta_1}(l_1),\widetilde{\Im}_{\beta_1}(l_2)\right\},\widetilde{\delta_1}\} \\ \min\{\hbar_{\beta_1}(l_1l_2),\gamma_2\} &\leq \max\{\min\{\hbar_{\beta_1}(l_1),\hbar_{\beta_1}(l_2)\},\delta_2\}. \end{aligned}$$

and

$$\begin{aligned} rmax\left\{\widetilde{\Im}_{\beta_1}(l_1l_2l_3),\widetilde{\gamma_1}\right\} &\succeq rmin\{\widetilde{\Im}_{\beta_1}(l_2),\widetilde{\delta_1}\} \\ \min\{\hbar_{\beta_1}(l_1l_2l_3),\gamma_2\} &\leq \max\{\hbar_{\beta_1}(l_2),\delta_2\}, \end{aligned}$$

where $\widetilde{\delta_1}$ *and* $\widetilde{\gamma_1}\in D(0,1]$ *such that* $\widetilde{\gamma_1}<\widetilde{\delta_1}$, *and* $\delta_2,\gamma_2\in[0,1]$ *such that* $\delta_2<\gamma_2$.

Proof. It follows from the proof of Theorem 3.2. □

4 Intra-regular $\mathcal{AG}$-groupoids

This is the principle segment and we we characterize Intra-regular $\mathcal{AG}$-groupoids with the assistance of differenet sorts of $(\in_\Gamma,\in_\Gamma\vee \boldsymbol{q}\boldsymbol{\Delta})$-cubic ideals of S.

Lemma 4.1. *Let* $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ *be a cubic set of an intra-regular* $\mathcal{AG}$*-groupoid* S*, then*

$$\mathcal{S}\circ\beta_1=_{(\Gamma,\Delta)}\beta_1,$$

and

$$\beta_1\circ\mathcal{S}=_{(\Gamma,\Delta)}\beta_1,$$

hold where $S=\left\langle\widetilde{\Im}_{\mathcal{S}},\hbar_{\mathcal{S}}\right\rangle=\langle\widetilde{1},0\rangle$.

Proof. Since S is intra-regular and let $g_1 \in S$, then there exist $l_1, l_2 \in S$ such that $g_1 = (l_1 g_1^2) l_2$. Now, $g_1 = (l_1(g_1 g_1)) l_2 = (g_1(l_1 g_1)) l_2 = (l_2(l_1 g_1)) g_1$. Therefore, we consider

$$\begin{aligned}
\widetilde{\Im}_{\mathcal{S}\circ\beta_1}(g_1) &= \underset{g_1=(l_2(l_1g_1))g_1}{\text{rsup}} \{r\min\{\widetilde{\Im}_{\mathcal{S}}(l_2(l_1g_1)), \widetilde{\Im}_{\beta_1}(g_1)\}\} \\
&= \underset{g_1=(l_2(l_1g_1))g_1}{\text{rsup}} \{r\min\{\widetilde{1}, \widetilde{\Im}_{\beta_1}(g_1)\}\} \\
&= \underset{g_1=(l_2(l_1g_1))g_1}{\text{rsup}} \{\widetilde{\Im}_{\beta_1}(g_1)\} \\
&= \widetilde{\Im}_{\beta_1}(g_1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\hbar_{\mathcal{S}\circ\beta_1}(g_1) &= \inf_{g_1=(l_2(l_1g_1))g_1} \{\max\{\hbar_{\mathcal{S}}(l_2(l_1g_1)), \hbar_{\beta_1}(g_1)\}\} \\
&= \inf_{g_1=(l_2(l_1g_1))g_1} \{\max\{0, \hbar_{\beta_1}(g_1)\}\} \\
&= \inf_{g_1=(l_2(l_1g_1))g_1} \{\hbar_{\beta_1}(g_1)\} \\
&= \hbar_{\beta_1}(g_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
rmax\{rmin\{\widetilde{\Im}_{\mathcal{S}\circ\beta_1}(g_1), \widetilde{\delta_1}\}, \widetilde{\gamma_1}\} &= rmax\{rmin\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{\delta_1}\}, \widetilde{\gamma_1}\} \\
\min\{\max\{\hbar_{\mathcal{S}\circ\beta_1}(g_1), \delta_2\}, \gamma_2\} &= \min\{\max\{\hbar_{\beta_1}(g_1), \delta_2\}, \gamma_2\}.
\end{aligned}$$

Thus, S$\circ\beta_1 =_{(\Gamma,\Delta)} \beta_1$ holds.

Now, for $\beta_1\circ$S$=_{(\Gamma,\Delta)} \beta_1$ we have

$$\begin{aligned}
g_1 &= (l_1 g_1^2) l_2 = (l_1 g_1^2)(g_5 l_2) \text{ as } g_5 \text{ is left identity} \\
&= (l_2 g_5)(g_1^2 l_1) \text{ by paramedial law} \\
&= (g_1 g_1)((l_2 g_5) l_1) \text{ by } g_1(g_2 g_3) = g_2(g_1 g_3) \\
&= (l_1(l_2 g_5))(g_1 g_1) \text{ by paramedial law} \\
&= g_1((l_1(l_2 g_5)) g_1) \text{ by } g_1(g_2 g_3) = g_2(g_1 g_3).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\widetilde{\Im}_{\beta_1\circ\mathcal{S}}(g_1) &= \underset{g_1=g_1((l_1(l_2g_5))g_1)}{\text{rsup}} \{r\min\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{\Im}_{\mathcal{S}}((l_1(l_2g_5))g_1)\}\} \\
&= \underset{g_1=g_1((l_1(l_2g_5))g_1)}{\text{rsup}} \{r\min\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{1}\}\} \\
&= \underset{g_1=g_1((l_1(l_2g_5))g_1)}{\text{rsup}} \{\widetilde{\Im}_{\beta_1}(g_1)\} \\
&= \widetilde{\Im}_{\beta_1}(g_1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\hbar_{\beta_1\circ\mathcal{S}}(g_1) &= \inf_{g_1=g_1((l_1(l_2g_5))g_1)} \{\max\{\hbar_{\beta_1}(g_1), \hbar_{\mathcal{S}}((l_1(l_2g_5))g_1)\}\} \\
&= \inf_{g_1=g_1((l_1(l_2g_5))g_1)} \{\max\{\hbar_{\beta_1}(g_1), 0\}\} \\
&= \inf_{g_1=g_1((l_1(l_2g_5))g_1)} \{\hbar_{\beta_1}(g_1)\} \\
&= \hbar_{\beta_1}(g_1).
\end{aligned}$$

Therefore,

$$\begin{aligned} rmax\{rmin\{\widetilde{\Im}_{\beta_1 \circ \mathcal{S}}(g_1), \widetilde{\delta_1}\}, \widetilde{\gamma_1}\} &= rmax\{rmin\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{\delta_1}\}, \widetilde{\gamma_1}\} \\ \min\left\{\max\left\{\hbar_{\beta_1 \circ \mathcal{S}}(g_1), \delta_2\right\}, \gamma_2\right\} &= \min\left\{\max\left\{\hbar_{\beta_1}(g_1), \delta_2\right\}, \gamma_2\right\}. \end{aligned}$$

Thus, $\beta_1 \circ \text{S} =_{(\Gamma,\Delta)} \beta_1$ holds. □

Corollary 4.1. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be a cubic left(resp,. right, two sided) ideal of an intra-regular* $\mathcal{AG}$*-groupoid* S*, then* $S \circ \beta_1 =_{(\Gamma,\Delta)} \beta_1$ *and* $\beta_1 \circ S =_{(\Gamma,\Delta)} \beta_1$ *hold where* $S = \left\langle \widetilde{\Im}_{\mathcal{S}}, \hbar_{\mathcal{S}} \right\rangle = \langle \widetilde{1}, 0 \rangle$.

Proof. It follows from the proof of Lemma 4.1. □

Theorem 4.1. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be a cubic set of an intra-regular* $\mathcal{AG}$*-groupoid* S *with left identity, then the following assertion are equivalent.*

(i) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic ideal of* S*.*

(ii) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic interior-ideal of* S*.*

Proof. It is obvious. □

Theorem 4.2. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be a cubic set of an intra-regular* $\mathcal{AG}$*-groupoid* S *with left identity, then the following conditions are equivalent.*

(i) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic left-ideal of* S*.*

(ii) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic right-ideal of* S*.*

(iii) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic ideal of* S*.*

(iv) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic bi-ideal of* S*.*

(v) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic generalized bi-ideal of* S*.*

(vi) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic interior-ideal of* S*.*

(vii) $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *is an* $(\in_\Gamma, \in_\Gamma \vee q\Delta)$*-cubic quasi-ideal of* S*.*

(viii) $\beta_1 \circ S =_{(\Gamma,\Delta)} \beta_1$ *and* $S \circ \beta_1 =_{(\Gamma,\Delta)} \beta_1$ *where* $S = \left\langle \widetilde{\Im}_{\mathcal{S}}, \hbar_{\mathcal{S}} \right\rangle = \langle \widetilde{1}, 0 \rangle$.

Proof. $(i) \Rightarrow (viii)$
It directly follows from the Corollary 4.1.

$(viii) \Rightarrow (vii)$ is obvious.

$(vii) \Rightarrow (vi)$
Let $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ be an $(\in_\Gamma, \in_\Gamma \vee q\boldsymbol{\Delta})$-cubic quasi-ideal of an intra-regular $\mathcal{AG}$-groupoid S with left identity. Since S is intra-regular and let $g_1 \in S$, then there exist $g_2, g_3 \in S$ such that $g_1 = (g_2 g_1^2) g_3$. Now, consider

$$\begin{aligned} (l_1 g_1) l_2 &= (l_1((g_2 g_1^2) g_3)) l_2 \text{ as } g_1 = (g_2 g_1^2) g_3 \\ &= ((g_2 g_1^2)(l_1 g_3)) l_2 \text{ by } g_1(g_2 g_3) = g_2(g_1 g_3) \\ &= ((g_3 l_1)(g_1^2 g_2)) l_2 \text{ by } (g_1 g_2)(g_3 g_4) = (g_4 g_3)(g_2 g_1) \\ &= (g_1^2((g_3 l_1) g_2)) l_2 \text{ by } g_1(g_2 g_3) = g_2(g_1 g_3) \\ &= (l_2((g_3 l_1) g_2))(g_1 g_1) \text{ by left invertive law} \\ &= g_1((l_2((g_3 l_1) g_2)) g_1) \text{ by } g_1(g_2 g_3) = g_2(g_1 g_3), \end{aligned}$$

and

$$\begin{aligned} (l_1 g_1) l_2 &= (l_1((g_2 g_1^2) g_3)) l_2 \text{ as } g_1 = (g_2 g_1^2) g_3 \\ &= ((g_2 g_1^2)(l_1 g_3)) l_2 \text{ by } g_1(g_2 g_3) = g_2(g_1 g_3) \\ &= ((g_3 l_1)(g_1^2 g_2)) l_2 \text{ by } (g_1 g_2)(g_3 g_4) = (g_4 g_3)(g_2 g_1) \\ &= (g_1^2((g_3 l_1) g_2)) l_2 \text{ by } g_1(g_2 g_3) = g_2(g_1 g_3) \\ &= (l_2((g_3 l_1) g_2))(g_1 g_1) \text{ by left invertive law} \\ &= (g_1 g_1)(((g_3 l_1) g_2) l_2) \text{ by } (g_1 g_2)(g_3 g_4) = (g_4 g_3)(g_2 g_1) \\ &= ((((g_3 l_1) g_2) l_2) g_1) g_1 \text{ by left invertive law.} \end{aligned}$$

Since $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ is an $(\in_\Gamma, \in_\Gamma \vee q\boldsymbol{\Delta})$-cubic quasi-ideal of S, thus,

$$rmax\left\{\widetilde{\Im}_{\beta_1}((l_1 g_1) l_2), \widetilde{\gamma}_1\right\} \succeq rmin\{\left(\widetilde{\Im}_{\beta_1} \circ \widetilde{\Im}_{\mathcal{S}}\right)((l_1 g_1) l_2), (\widetilde{\Im}_{\mathcal{S}} \circ \widetilde{\Im}_{\beta_1})((l_1 g_1) l_2), \widetilde{\delta_1}\}, \tag{1}$$

and

$$\min\{\hbar_{\beta_1}((l_1 g_1) l_2), \gamma_2\} \leq \max\{(\hbar_{\beta_1} \circ \hbar_{\mathcal{S}})((l_1 g_1) l_2), (\hbar_{\mathcal{S}} \circ \hbar_{\beta_1})((l_1 g_1) l_2), \delta_2\}, \tag{2}$$

where S= $\left\langle \widetilde{\Im}_{\mathcal{S}}, \hbar_{\mathcal{S}} \right\rangle = \langle \widetilde{1}, 0 \rangle$. We consider

$$\begin{aligned} &\left(\widetilde{\Im}_{\beta_1} \circ \widetilde{\Im}_{\mathcal{L}}\right)((l_1 g_1) l_2) \\ &= r \sup_{(l_1 g_1) l_2 = g_1((l_2((g_3 l_1) g_2)) g_1)} \{r \min\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{\Im}_{\mathcal{L}}(((l_2((g_3 l_1) g_2)) g_1))\}\} \\ &= r \sup_{(l_1 g_1) l_2 = g_1((l_2((g_3 l_1) g_2)) g_1)} \{r \min\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{1}\}\} \succeq \widetilde{\Im}_{\beta_1}(g_1), \end{aligned} \tag{3}$$

and

$$\begin{aligned} &(\widetilde{\Im}_{\mathcal{L}} \circ \widetilde{\Im}_{\beta_1})((l_1 g_1) l_2) \\ &= r \sup_{(l_1 g_1) l_2 = ((((g_3 l_1) g_2) l_2) g_1) g_1} \{r \min\{\widetilde{\Im}_{\mathcal{L}}(((((g_3 l_1) g_2) l_2) g_1)), \widetilde{\Im}_{\beta_1}(g_1)\}\} \\ &= r \sup_{(l_1 g_1) l_2 = ((((g_3 l_1) g_2) l_2) g_1) g_1} \{r \min\{\widetilde{1}, \widetilde{\Im}_{\beta_1}(g_1)\}\} \succeq \widetilde{\Im}_{\beta_1}(g_1). \end{aligned} \tag{4}$$

By using (3) and (4) into (1), we get

$$\begin{aligned} &\text{rmax}\left\{\widetilde{\Im}_{\beta_1}((l_1 g_1) l_2), \widetilde{\gamma}_1\right\} \\ &\succeq \text{rmin}\{\left(\widetilde{\Im}_{\beta_1} \circ \widetilde{\Im}_{\mathcal{L}}\right)((l_1 g_1) l_2), (\widetilde{\Im}_{\mathcal{L}} \circ \widetilde{\Im}_{\beta_1})((l_1 g_1) l_2), \widetilde{\delta_1}\} \\ &\succeq \text{rmin}\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{\Im}_{\beta_1}(g_1), \widetilde{\delta_1}\} = \text{rmin}\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{\delta_1}\}. \end{aligned} \tag{5}$$

On the other hand,

$$\begin{aligned}&(\hbar_{\beta_1}\circ\hbar_{\mathcal{L}})\,((l_1g_1)l_2)\\&=\inf_{(l_1g_1)l_2=g_1((l_2((g_3l_1)g_2))g_1)}\{\max\{\hbar_{\beta_1}(g_1),\hbar_{\mathcal{L}}(((l_2((g_3l_1)g_2))g_1))\}\}\\&=\inf_{(l_1g_1)l_2=g_1((l_2((g_3l_1)g_2))g_1)}\{\max\{\hbar_{\beta_1}(g_1),0\}\}\leq\hbar_{\beta_1}(g_1),\end{aligned}\tag{6}$$

and

$$\begin{aligned}&(\hbar_{\mathcal{L}}\circ\hbar_{\beta_1})\,((l_1g_1)l_2)\\&=\inf_{(l_1g_1)l_2=((((g_3l_1)g_2)l_2)g_1)g_1}\{\max\{\hbar_{\mathcal{L}}(((((g_3l_1)g_2)l_2)g_1)),\hbar_{\beta_1}(g_1)\}\}\\&=\inf_{(l_1g_1)l_2=((((g_3l_1)g_2)l_2)g_1)g_1}\{\max\{0,\hbar_{\beta_1}(g_1)\}\}\leq\hbar_{\beta_1}(g_1).\end{aligned}\tag{7}$$

By using (6) and (7) into (2), we get

$$\begin{aligned}&\min\left\{\hbar_{\beta_1}\left((l_1g_1)l_2\right),\gamma_2\right\}\\&\leq\max\left\{(\hbar_{\beta_1}\circ\hbar_{\mathcal{L}})\left((l_1g_1)l_2\right),(\hbar_{\mathcal{L}}\circ\hbar_{\beta_1})\left((l_1g_1)l_2\right),\delta_2\right\}\\&\leq\max\left\{\hbar_{\beta_1}(g_1),\hbar_{\beta_1}\left(g_1\right),\delta_2\right\}=\max\left\{\hbar_{\beta_1}(g_1),\delta_2\right\}.\end{aligned}\tag{8}$$

From (5) and (8), we have $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ is an $(\in_\Gamma,\in_\Gamma\vee\boldsymbol{q}\boldsymbol{\Delta})$-cubic interior-ideal of S.

$(vi)\Rightarrow(v)$
Let $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ be an $(\in_\Gamma,\in_\Gamma\vee\boldsymbol{q}\boldsymbol{\Delta})$-cubic interior-ideal of an intra-regular $\mathcal{AG}$-groupoid S with left identity. Then by Theorem 4.1, $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ is an $(\in_\Gamma,\in_\Gamma\vee\boldsymbol{q}\boldsymbol{\Delta})$-cubic ideal of S. So it is obviously an $(\in_\Gamma,\in_\Gamma\vee\boldsymbol{q}\boldsymbol{\Delta})$-cubic generalized bi-ideal.

$(v)\Rightarrow(iv)$ is obvious.

$(iv)\Rightarrow(iii)$
Let $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ be an $(\in_\Gamma,\in_\Gamma\vee\boldsymbol{q}\boldsymbol{\Delta})$-cubic bi-ideal of an intra-regular $\mathcal{AG}$-groupoid S with left identity. Since S is intra-regular and let $g_1\in S$, then there exist $l_1,l_2\in S$ such that $g_1=(l_1g_1^2)l_2$.

Now consider,

$$\begin{aligned}(g_1g_2)&=(((l_1(g_1g_1))l_2)g_2)=(((g_1(l_1g_1))l_2)g_2)\\&=((g_2l_2)((g_5g_1)(l_1g_1)))=((g_2l_2)((g_1l_1)(g_1g_5)))\\&=(((g_1g_5)(g_1l_1))(l_2g_2))=((g_1((g_1g_5)l_1))(l_2g_2))\\&=(((l_2g_2)((g_1g_5)l_1))g_1)=(((l_2g_2)((((l_1g_1^2)l_2)g_5)l_1))g_1)\\&=(((l_2g_2)((l_2(l_1g_1^2))(g_5l_1)))g_1)=(((l_2g_2)((l_1g_5)((l_1g_1^2)(g_5l_2))))g_1)\\&=(((l_2g_2)((l_1g_5)((l_2g_5)(g_1^2l_1))))g_1)=(((l_2g_2)((l_1g_5)(g_1^2((l_2g_5)l_1))))g_1)\\&=(((l_2g_2)(g_1^2((l_1g_5)((l_2g_5)l_1))))g_1)=((g_1^2((l_2g_2)((l_1g_5)((l_2g_5)l_1))))g_1).\end{aligned}$$

Now,

$$\begin{aligned}r\max\{\widetilde{\Im}_{\beta_1}(g_1g_2),\widetilde{\gamma_1}\}&=r\max\{\widetilde{\Im}_{\beta_1}(((g_1^2((l_2g_2)((l_1g_5)((l_2g_5)l_1))))g_1)),\widetilde{\gamma_1}\}\\&\succeq r\min\{\widetilde{\Im}_{\beta_1}(g_1^2),\widetilde{\Im}_{\beta_1}(g_1),\widetilde{\delta_1}\}\\&=r\min\{\widetilde{\Im}_{\beta_1}(g_1),\widetilde{\Im}_{\beta_1}(g_1),\widetilde{\delta_1}\}=r\min\{\widetilde{\Im}_{\beta_1}(g_1),\widetilde{\delta_1}\}\end{aligned}$$

and

$$\begin{aligned}\min\{\hbar_{\beta_1}(g_1g_2),\gamma_2\} &= \min\{\hbar_{\beta_1}(((g_1^2((l_2g_2)((l_1g_5)((l_2g_5)l_1))))g_1)),\gamma_2\}\\ &\leq \max\{\hbar_{\beta_1}(g_1^2),\hbar_{\beta_1}(g_1),\delta_2\}\\ &= \max\{\hbar_{\beta_1}(g_1),\hbar_{\beta_1}(g_1),\delta_2\} = \max\{\hbar_{\beta_1}(g_1),\delta_2\}.\end{aligned}$$

Thus, $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ is an $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic right ideal of S, which is also an $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic left ideal of S. Hence $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ is an $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic ideal of S.

$(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious. □

Definition 4.1. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *be two cubic sets of* S. *We define the cubic sets* $\beta_1^{*} = \left\langle \widetilde{\Im}_{\beta_1^{*}}, \hbar_{\beta_1^{*}} \right\rangle$, $\beta_1 \wedge^{*} \beta_2 = \left\langle \widetilde{\Im}_{\beta_1\wedge^{*}\beta_2}, \hbar_{\beta_1\vee^{*}\beta_2} \right\rangle$, $\beta_1 \vee^{*} \beta_2 = \left\langle \widetilde{\Im}_{\beta_1\vee^{*}\beta_2}, \hbar_{\beta_1\wedge^{*}\beta_2} \right\rangle$ *and* $\beta_1 \circ^{*} \beta_2 = \left\langle \widetilde{\Im}_{\beta_1\circ^{*}\beta_2}, \hbar_{\beta_1\circ^{*}\beta_2} \right\rangle$ *as follows:*

(i)

$$\begin{aligned}&\beta_1^{*}(g_1)\\ &= \left\langle \widetilde{\Im}_{\beta_1^{*}}(g_1), \hbar_{\beta_1^{*}}(g_1) \right\rangle\\ &= \left\langle r\min\{r\max\{\widetilde{\Im}_{\beta_1}(g_1), \widetilde{\gamma_1}\}, \widetilde{\delta_1}\}, \max\{\min\{\hbar_{\beta_1}(g_1),\gamma_2\},\delta_2\} \right\rangle\\ &= \left\langle (\widetilde{\Im}_{\beta_1}(g_1)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}, (\hbar_{\beta_1}(g_1)\wedge\gamma_2)\vee\delta_2 \right\rangle,\end{aligned}$$

(ii)

$$\begin{aligned}&\beta_1 \wedge^{*} \beta_2(g_1)\\ &= \left\langle \widetilde{\Im}_{\beta_1\wedge^{*}\beta_2}(g_1), \hbar_{\beta_1\vee^{*}\beta_2}(g_1) \right\rangle\\ &= \left\langle \begin{array}{c} r\min\{r\max\{(\widetilde{\Im}_{\beta_1}\widetilde{\wedge}\widetilde{\Im}_{\beta_2})(g_1), \widetilde{\gamma_1}\}, \widetilde{\delta_1}\},\\ \max\{\min\{(\hbar_{\beta_1}\vee\hbar_{\beta_2})(g_1),\gamma_2\},\delta_2\} \end{array} \right\rangle\\ &= \left\langle ((\widetilde{\Im}_{\beta_1}\widetilde{\wedge}\widetilde{\Im}_{\beta_2})(g_1)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}, ((\hbar_{\beta_1}\vee\hbar_{\beta_2})(g_1)\wedge\gamma_2)\vee\delta_2 \right\rangle,\end{aligned}$$

(iii)

$$\begin{aligned}&\beta_1 \vee^{*} \beta_2(g_1)\\ &= \left\langle \widetilde{\Im}_{\beta_1\vee^{*}\beta_2}(g_1), \hbar_{\beta_1\wedge^{*}\beta_2}(g_1) \right\rangle\\ &= \left\langle \begin{array}{c} r\min\{r\max\{(\widetilde{\Im}_{\beta_1}\widetilde{\vee}\widetilde{\Im}_{\beta_2})(g_1), \widetilde{\gamma_1}\}, \widetilde{\delta_1}\},\\ \max\{\min\{(\hbar_{\beta_1}\wedge\hbar_{\beta_2})(g_1),\gamma_2\},\delta_2\} \end{array} \right\rangle\\ &= \left\langle ((\widetilde{\Im}_{\beta_1}\widetilde{\vee}\widetilde{\Im}_{\beta_2})(g_1)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}, ((\hbar_{\beta_1}\wedge\hbar_{\beta_2})(g_1)\wedge\gamma_2)\vee\delta_2 \right\rangle,\end{aligned}$$

(iv)

$$\begin{aligned}
&\beta_1 \circ^{\divideontimes} \beta_2(g_1)\\
&= \left\langle \widetilde{\Im}_{\beta_1 \circ^{\divideontimes} \beta_2}(g_1), \hbar_{\beta_1 \circ^{\divideontimes} \beta_2}(g_1) \right\rangle\\
&= \left\langle \begin{array}{c} r\min\{r\max\{(\widetilde{\Im}_{\beta_1} \widetilde{\circ} \widetilde{\Im}_{\beta_2})(g_1), \widetilde{\gamma_1}\}, \widetilde{\delta_1}\}, \\ \max\{\min\{(\hbar_{\beta_1} \circ \hbar_{\beta_2})(g_1), \gamma_2\}, \delta_2\} \end{array} \right\rangle\\
&= \left\langle ((\widetilde{\Im}_{\beta_1} \widetilde{\circ} \widetilde{\Im}_{\beta_2})(g_1) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1}, ((\hbar_{\beta_1} \circ \hbar_{\beta_2})(g_1) \wedge \gamma_2) \vee \delta_2 \right\rangle,
\end{aligned}$$

where $\widetilde{\delta_1}, \widetilde{\gamma_1} \in D(0,1]$ *such that* $\widetilde{\gamma_1} \prec \widetilde{\delta_1}$ *and* $\delta_2, \gamma_2 \in [0,1)$ *such that* $\delta_2 < \gamma_2$.

Here we prove a lemma which will be very helpful.

Lemma 4.2. *For and two cubic sets* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *of* S*, the following assertion are true,*

(*i*) $\beta_1 \wedge^{\divideontimes} \beta_2 = \beta_1^{\divideontimes} \wedge \beta_2^{\divideontimes}$.

(*ii*) $\beta_1 \vee^{\divideontimes} \beta_2 = \beta_1^{\divideontimes} \vee \beta_2^{\divideontimes}$.

(*iii*) $\beta_1 \circ^{\divideontimes} \beta_2 = \beta_1^{\divideontimes} \circ \beta_2^{\divideontimes}$.

Lemma 4.3. *Let* R_5 *and* R_4 *be any two non-empty subsets of* S*. Then the accompanying statements are valid,*

(*i*) $\varkappa_{\Gamma}^{\Delta} R_5 \wedge^{\divideontimes} \varkappa_{\Gamma}^{\Delta} R_4 = \varkappa_{\Gamma}^{\divideontimes\Delta} R_5 \cap R_4$,

(*ii*) $\varkappa_{\Gamma}^{\Delta} R_5 \vee^{\divideontimes} \varkappa_{\Gamma}^{\Delta} R_4 = \varkappa_{\Gamma}^{\divideontimes\Delta} R_5 \cup R_4$,

(*iii*) $\varkappa_{\Gamma}^{\Delta} R_5 \circ^{\divideontimes} \varkappa_{\Gamma}^{\Delta} R_4 = \varkappa_{\Gamma}^{\divideontimes\Delta} R_5 \circ R_4$.

Lemma 4.4. *Every* $(\in_{\Gamma}, \in_{\Gamma} \vee \boldsymbol{q\Delta})$*-cubic ideal* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *of* S *with left identity is idempotent.*

Proof. Let $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ be an $(\in_{\Gamma}, \in_{\Gamma} \vee \boldsymbol{q\Delta})$-cubic ideal of an intra-regular $\mathcal{AG}$-groupoid S with left identity. Since S is intra-regular so for each $g_1 \in S$ there exist $l_1, l_2 \in S$ such that $g_1 = (l_1 g_1^2) l_2$. Now, as

$$g_1 = (l_1 g_1^2) l_2 = (l_1(g_1 g_1)) l_2 = (g_1(l_1 g_1)) l_2 = g_1^2(l_1 l_2) = (l_2(l_1 g_1)) g_1.$$

Consider,

$$\begin{aligned}
\widetilde{\Im}_{\beta_1 \circ^* \beta_1}(g_1) &= ((\widetilde{\Im}_{\beta_1} \widetilde{\circ} \widetilde{\Im}_{\beta_1})(g_1) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\
&= ((\operatorname*{rsup}_{g_1 = p_1 \circ p_2} \{ \widetilde{\Im}_{\beta_1}(p_1) \widetilde{\wedge} \widetilde{\Im}_{\beta_1}(p_2) \}) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\
&\succeq (\{ \widetilde{\Im}_{\beta_1}(l_2(l_1 g_1)) \widetilde{\wedge} \widetilde{\Im}_{\beta_1}(g_1) \} \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\
&= \{ (\widetilde{\Im}_{\beta_1}(l_2(l_1 g_1)) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} (\widetilde{\Im}_{\beta_1}(g_1) \widetilde{\vee} \widetilde{\gamma_1}) \} \widetilde{\wedge} \widetilde{\delta_1} \\
&\succeq (\{ (\widetilde{\Im}_{\beta_1}(g_1) \widetilde{\wedge} \widetilde{\delta_1}) \widetilde{\wedge} (\widetilde{\Im}_{\beta_1}(g_1) \widetilde{\wedge} \widetilde{\delta_1}) \} \widetilde{\wedge} \widetilde{\delta_1}) \widetilde{\vee} \widetilde{\gamma_1} \\
&= (((\widetilde{\Im}_{\beta_1}(g_1) \widetilde{\wedge} \widetilde{\Im}_{\beta_1}(g_1)) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\
&= \widetilde{\Im}_{\beta_1^*} \widetilde{\wedge} \widetilde{\Im}_{\beta_1^*}(g_1) \\
&= \widetilde{\Im}_{\beta_1^*}(g_1),
\end{aligned}$$

and

$$\begin{aligned}
\hbar_{\beta_1 \circ^* \beta_1}(g_1) &= ((\hbar_{\beta_1} \circ \hbar_{\beta_1})(g_1) \wedge \gamma_2) \vee \delta_2 \\
&= ((\inf_{g_1 = p_1 \circ p_2} \{ \hbar_{\beta_1}(p_1) \vee \hbar_{\beta_1}(p_2) \}) \wedge \gamma_2) \vee \delta_2 \\
&\leq \{ (\hbar_{\beta_1}(l_2(l_1 g_1) \vee \hbar_{\beta_1}(g_1)) \wedge \gamma_2) \vee \delta_2 \\
&= \{ (\hbar_{\beta_1}(g_1(vu) \wedge \gamma_2) \vee (\hbar_{\beta_1}(g_1 l_1) \wedge \gamma_2) \vee \delta_2 \\
&\leq (\hbar_{\beta_1}(g_1) \vee \delta_2) \vee (\hbar_{\beta_1}(g_1) \vee \delta_2) \wedge \gamma_2 \\
&= ((\hbar_{\beta_1}(g_1) \vee \hbar_{\beta_1}(g_1)) \wedge \gamma_2) \vee \delta_2 \\
&= (((\hbar_{\beta_1} \vee \hbar_{\beta_1})(g_1)) \wedge \gamma_2) \vee \delta_2 \\
&= \left(\hbar_{\beta_1^*} \vee \hbar_{\beta_1^*} \right)(g_1) \\
&= \hbar_{\beta_1^*}(g_1).
\end{aligned}$$

So, $\beta_1 \circ^* \beta_1(g_1) = \left\langle \widetilde{\Im}_{\beta_1 \circ^* \beta_1}(g_1) \succeq (\widetilde{\Im}_{\beta_1^*}(g_1), \hbar_{\beta_1 \circ^* \beta_1}(g_1) \leq \hbar_{\beta_1^*}(g_1) \right\rangle \supseteq \beta_1$.

Also,

$$\begin{aligned}
\widetilde{\Im}_{\beta_1 \circ^* \beta_1}(g_1) &= ((\widetilde{\Im}_{\beta_1} \widetilde{\circ} \widetilde{\Im}_{\beta_1})(g_1) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\
&= ((\operatorname*{rsup}_{g_1 = p_1 \circ p_2} \{ \widetilde{\Im}_{\beta_1}(p_1) \widetilde{\wedge} \widetilde{\Im}_{\beta_1}(p_2) \}) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\
&\preceq \operatorname*{rsup}_{g_1 = p_1 \circ p_2} (\widetilde{\Im}_{\beta_1}(p_1 p_2) \widetilde{\wedge} \widetilde{\delta_1}) \widetilde{\wedge} (\widetilde{\Im}_{\beta_1}(p_1 p_2) \widetilde{\wedge} \widetilde{\delta_1}) \widetilde{\vee} \widetilde{\gamma_1} \\
&= \widetilde{\Im}_{\beta_1^*} \widetilde{\wedge} \widetilde{\Im}_{\beta_1^*}(g_1) \\
&= \widetilde{\Im}_{\beta_1^*}(g_1),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
\hbar_{\beta_1 \circ^* \beta_1}(g_1) &= ((\hbar_{\beta_1} \circ \hbar_{\beta_1})(g_1) \wedge \gamma_2) \vee \delta_2 \\
&= ((\inf_{g_1 = p_1 \circ p_2} \{ \hbar_{\beta_1}(p_1) \vee \hbar_{\beta_1}(p_2) \}) \wedge \gamma_2) \vee \delta_2 \\
&\geq \inf_{g_1 = p_1 \circ p_2} (\hbar_{\beta_1}(p_1 p_2) \vee \delta_2) \vee (\hbar_{\beta_1}(p_1 p_2) \vee \delta_2) \wedge \gamma_2 \\
&= \hbar_{\beta_1^*} \vee \hbar_{\beta_1^*}(g_1) \\
&= \hbar_{\beta_1^*}(g_1).
\end{aligned}$$

So, $\beta_1 \circ^* \beta_1(g_1) = \left\langle \widetilde{\Im}_{\beta_1 \circ^* \beta_1}(g_1) \preceq \left(\widetilde{\Im}_{\beta_1^*} \right)(g_1), \hbar_{\beta_1 \circ^* \beta_1}(g_1) \geq \left(\hbar_{\beta_1^*} \right)(g_1) \right\rangle \subseteq \beta_1$. Hence, $\beta_1 = \beta_1 \circ^* \beta_1$. □

Theorem 4.3. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be a cubic set of* S *with left identity, the accompanying conditions are comparable,*

(i) *S is intra-regular.*

(ii) $\beta_1 \wedge^{*} \beta_2 = \beta_1 \circ^{*} \beta_2$ *for all* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic quasi ideals* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *of S.*

Proof. $(i) \Rightarrow (ii)$
Let $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ be $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic quasi ideals of an intra-regular $\mathcal{AG}$-groupoid S with left identity. Then by Theorem 4.2, $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ become $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic ideals of S. Since S is intra-regular so for each $g_1 \in S$ there exist $l_1, l_2 \in S$ such that $g_1 = (l_1 g_1^2) l_2$. Now, as

$$g_1 = (l_1 g_1^2) l_2 = (l_1(g_1 g_1)) l_2 = (g_1(l_1 g_1)) l_2 = g_1^2((vu) l_1) = (l_2(l_1 g_1)) g_1.$$

As $\beta_1 \circ^{*} \beta_2(g_1) = \left\langle \widetilde{\Im}_{\beta_1 \circ^{*} \beta_2}(g_1), \hbar_{\beta_1 \circ^{*} \beta_2}(g_1) \right\rangle$.

Consider first,

$$\begin{aligned}
\widetilde{\Im}_{\beta_1 \circ^{*} \beta_2}(g_1) &= ((\widetilde{\Im}_{\beta_1} \widetilde{\circ} \widetilde{\Im}_{\beta_2})(g_1) \widetilde{\vee} \widetilde{\gamma}_1) \widetilde{\wedge} \widetilde{\delta}_1 \\
&= ((\operatorname*{rsup}_{g_1 = p_1 \circ p_2} \{\widetilde{\Im}_{\beta_1}(p_1) \widetilde{\wedge} \widetilde{\Im}_{\beta_2}(p_2)\}) \widetilde{\vee} \widetilde{\gamma}_1) \widetilde{\wedge} \widetilde{\delta}_1 \\
&\succeq (\{\widetilde{\Im}_{\beta_1}(l_2(l_1 g_1)) \widetilde{\wedge} \widetilde{\Im}_{\beta_2}(g_1)\} \widetilde{\vee} \widetilde{\gamma}_1) \widetilde{\wedge} \widetilde{\delta}_1 \\
&= \{(\widetilde{\Im}_{\beta_1}(l_2(l_1 g_1)) \widetilde{\vee} \widetilde{\gamma}_1) \widetilde{\wedge} (\widetilde{\Im}_{\beta_2}(g_1) \widetilde{\vee} \widetilde{\gamma}_1)\} \widetilde{\wedge} \widetilde{\delta}_1 \\
&\succeq (\{(\widetilde{\Im}_{\beta_1}(g_1) \widetilde{\wedge} \widetilde{\delta}_1) \widetilde{\wedge} (\widetilde{\Im}_{\beta_2}(g_1) \widetilde{\wedge} \widetilde{\delta}_1)\} \widetilde{\wedge} \widetilde{\delta}_1) \widetilde{\vee} \widetilde{\gamma}_1 \\
&= (((\widetilde{\Im}_{\beta_1}(g_1) \widetilde{\wedge} \widetilde{\Im}_{\beta_2}(g_1)) \widetilde{\vee} \widetilde{\gamma}_1) \widetilde{\wedge} \widetilde{\delta}_1 \\
&= \widetilde{\Im}_{\beta_1^{*}} \widetilde{\wedge} \widetilde{\Im}_{\beta_2^{*}}(g_1),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
\hbar_{\beta_1 \circ^{*} \beta_2}(g_1) &= ((\hbar_{\beta_1} \circ \hbar_{\beta_2})(g_1) \wedge \gamma_2) \vee \delta_2 \\
&= ((\inf_{g_1 = p_1 \circ p_2} \{\hbar_{\beta_1}(p_1) \vee \hbar_{\beta_2}(p_2)\}) \wedge \gamma_2) \vee \delta_2 \\
&\leq \{(\hbar_{\beta_1}(l_2(l_1 g_1) \vee \hbar_{\beta_2}(g_1)) \wedge \gamma_2) \vee \delta_2 \\
&= \{(\hbar_{\beta_1}(g_1(vu) \wedge \gamma_2) \vee (\hbar_{\beta_2}(g_1 l_1) \wedge \gamma_2) \vee \delta_2 \\
&\leq (\hbar_{\beta_1}(g_1) \vee \delta_2) \vee (\hbar_{\beta_2}(g_1) \vee \delta_2) \wedge \gamma_2 \\
&= ((\hbar_{\beta_1}(g_1) \vee \hbar_{\beta_2}(g_1)) \wedge \gamma_2) \vee \delta_2 \\
&= (((\hbar_{\beta_1} \vee \hbar_{\beta_2})(g_1)) \wedge \gamma_2) \vee \delta_2 \\
&= \left(\hbar_{\beta_1^{*}} \vee \hbar_{\beta_2^{*}}\right)(g_1).
\end{aligned}$$

So,

$$\begin{aligned}
&\beta_1 \circ^{*} \beta_2(g_1) \\
&= \left\langle \widetilde{\Im}_{\beta_1 \circ^{*} \beta_2}(g_1) \succeq (\widetilde{\Im}_{\beta_1^{*}} \widetilde{\wedge} \widetilde{\Im}_{\beta_2^{*}})(g_1), \hbar_{\beta_1 \circ^{*} \beta_2}(g_1) \leq \left(\hbar_{\beta_1^{*}} \vee \hbar_{\beta_2^{*}}\right)(g_1) \right\rangle \\
&\supseteq \beta_1 \wedge^{*} \beta_2.
\end{aligned}$$

Also,

$$\begin{aligned}\widetilde{\Im}_{\beta_1\circ^*\beta_2}(g_1) &= ((\widetilde{\Im}_{\beta_1}\widetilde{\circ}\widetilde{\Im}_{\beta_2})(g_1)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\ &= ((\operatorname*{rsup}_{g_1=p_1\circ p_2}\{\widetilde{\Im}_{\beta_1}(p_1)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(p_2)\})\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\ &\preceq \operatorname*{rsup}_{g_1=p_1\circ p_2}(\widetilde{\Im}_{\beta_1}(p_1p_2)\widetilde{\wedge}\widetilde{\delta_1})\widetilde{\wedge}(\widetilde{\Im}_{\beta_2}(p_1p_2)\widetilde{\wedge}\widetilde{\delta_1})\widetilde{\vee}\widetilde{\gamma_1}\\ &= \widetilde{\Im}_{\beta_1^*}\widetilde{\wedge}\widetilde{\Im}_{\beta_2^*}(g_1),\end{aligned}$$

and on the other hand,

$$\begin{aligned}\hbar_{\beta_1\circ^*\beta_2}(g_1) &= ((\hbar_{\beta_1}\circ\hbar_{\beta_2})(g_1)\wedge\gamma_2)\vee\delta_2\\ &= ((\inf_{g_1=p_1\circ p_2}\{\hbar_{\beta_1}(p_1)\vee\hbar_{\beta_2}(p_2)\})\wedge\gamma_2)\vee\delta_2\\ &\geq \inf_{g_1=p_1\circ p_2}(\hbar_{\beta_1}(p_1p_2)\vee\delta_2)\vee(\hbar_{\beta_2}(p_1p_2)\vee\delta_2)\wedge\gamma_2\\ &= \hbar_{\beta_1^*}\vee\hbar_{\beta_2^*}(g_1).\end{aligned}$$

So,

$$\begin{aligned}&\beta_1\circ^*\beta_2(g_1)\\ &=\left\langle\widetilde{\Im}_{\beta_1\circ^*\beta_2}(g_1)\preceq\left(\widetilde{\Im}_{\beta_1^*}\widetilde{\wedge}\widetilde{\Im}_{\beta_2^*}\right)(g_1),\hbar_{\beta_1\circ^*\beta_2}(g_1)\geq\left(\hbar_{\beta_1^*}\vee\hbar_{\beta_2^*}\right)(g_1)\right\rangle\\ &\subseteq\beta_1\wedge^*\beta_2.\end{aligned}$$

Hence, $\beta_1\wedge^*\beta_2=\beta_1\circ^*\beta_2$.

(ii)⇒(i)
Let Q_1 and Q_2 are the quasi ideals of S with left identity and let $g_1\in Q_1\cap Q_2$. Then $\varkappa_\Gamma^{*\Delta}Q_1$ and $\varkappa_\Gamma^{*\Delta}Q_2$ are $(\in_\Gamma,\in_\Gamma\vee\boldsymbol{q\Delta})$-cubic quasi ideals of S. Then, by hypothesis

$$\varkappa_\Gamma^{*\Delta}Q_1Q_2=\varkappa_\Gamma^{\Delta}Q_1\circ^*\varkappa_\Gamma^{\Delta}Q_2=\varkappa_\Gamma^{\Delta}Q_1\wedge^*\varkappa_\Gamma^{\Delta}Q_2=\varkappa_\Gamma^{\Delta}Q_1\cap Q_2.$$

Thus, $Q_1Q_2=Q_1\cap Q_2$. Hence, S is intra-regular by Theorem 2.1. □

Theorem 4.4. *Let $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ be a cubic set of S with left identity, the accompanying conditions are comparable,*

(i) S is intra-regular.

(ii) $\beta_1\wedge^*\beta_2\subseteq\beta_1\circ^*\beta_2$ for all $(\in_\Gamma,\in_\Gamma\vee\boldsymbol{q\Delta})$-cubic left ideal $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ and every $(\in_\Gamma,\in_\Gamma\vee\boldsymbol{q\Delta})$-cubic right ideal $\beta_2=\left\langle\widetilde{\Im}_{\beta_2},\hbar_{\beta_2}\right\rangle$ of S.

Proof. Straightforward. □

Theorem 4.5. *Let $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ be a cubic set of S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $\beta_1 \wedge^* \beta_2 = \beta_1 \circ^* \beta_2$ *for any* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic quasi ideal* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and any* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic left ideal* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *of S.*

(iii) $\beta_1 \wedge^* \beta_2 = \beta_1 \circ^* \beta_2$ *for all* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic quasi ideals* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *of S.*

Proof. It follows from the proof of the Theorem 4.3. □

Theorem 4.6. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be a cubic set of S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $\beta_1 \wedge^* \beta_2 \subseteq \beta_1 \circ^* \beta_2$ *for any* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic quasi ideal* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and any* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic left ideal* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *of S.*

(iii) $\beta_1 \wedge^* \beta_2 \subseteq \beta_1 \circ^* \beta_2$ *for all* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic quasi ideals* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *of S.*

Proof. It follows from the proof of the Theorem 4.3. □

Theorem 4.7. *Let* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *be a cubic set of S with left identity, the following conditions are equivalent.*

(i) S is intra-regular.

(ii) $\beta_1 \wedge^* \beta_2 = (\beta_1 \circ^* \beta_2) \circ^* \beta_1$ *for any* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic left ideal* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and any* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic quasi ideal* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *of S.*

(iii) $\beta_1 \wedge^* \beta_2 = (\beta_1 \circ^* \beta_2) \circ^* \beta_1$ *for any* $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$*-cubic quasi ideals* $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ *and* $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ *of S.*

Proof. (i)⇒(iii)
Let $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ be $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic quasi ideals of an intra-regular $\mathcal{AG}$-groupoid S with left identity. Then, by Theorem 4.2, $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ become $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic ideals of S. Since S is intra-regular so for each $g_1 \in S$, there exist $l_1, l_2 \in S$ such that $g_1 = (l_1 g_1^2) l_2$.
Now as

$$\begin{aligned} g_1 &= (l_1 g_1^2) l_2 = (l_1(g_1 g_1)) l_2 = (g_1(l_1 g_1)) l_2 = g_1^2((vu) l_1) \\ &= (l_2(l_1 g_1)) g_1 = (l_2(l_1 g_1))((l_2(l_1 g_1)) g_1) = (g_1(l_2(l_1 g_1)))((l_1 g_1) l_2). \end{aligned}$$

Now, we consider

$$\begin{aligned}
\widetilde{\Im}_{(\beta_1\circ^*\beta_2)\circ^*\beta_1}(g_1) &= (((\widetilde{\Im}_{\beta_1}\widetilde{\circ}\widetilde{\Im}_{\beta_2})\widetilde{\circ}\widetilde{\Im}_{\beta_1})(g_1)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&= ((\operatorname*{rsup}_{g_1=p_1p_2}\{(\widetilde{\Im}_{\beta_1}\widetilde{\circ}\widetilde{\Im}_{\beta_2})(p_1)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(p_2)\})\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&\succeq (\{(\widetilde{\Im}_{\beta_1}\widetilde{\circ}\widetilde{\Im}_{\beta_2})(g_1(l_2(l_1g_1)))\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(g_1)\}\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&= \operatorname*{rsup}_{g_1(l_2(l_1g_1))=uv}\{\widetilde{\Im}_{\beta_1}(u)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(v)\}\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(g_1)\widetilde{\wedge}\widetilde{\delta_1}\\
&\succeq (\{(\widetilde{\Im}_{\beta_1}(g_1)\widetilde{\wedge}\widetilde{\delta_1})\widetilde{\wedge}(\widetilde{\Im}_{\beta_2}(g_1)\widetilde{\wedge}\widetilde{\delta_1})\}\widetilde{\Im}_{\beta_2}(g_1)\widetilde{\wedge}\widetilde{\delta_1})\widetilde{\vee}\widetilde{\gamma_1}\\
&= (((\widetilde{\Im}_{\beta_1}(g_1)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(g_1))\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&= \widetilde{\Im}_{\beta_1^*}\widetilde{\wedge}\widetilde{\Im}_{\beta_2^*}(g_1),
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\hbar_{(\beta_1\circ^*\beta_2)\circ^*\beta_1}(g_1) &= (((\hbar_{\beta_1}\circ\hbar_{\beta_2})\circ\hbar_{\beta_1})(g_1)\wedge\gamma_2)\vee\delta_2\\
&= ((\inf_{g_1=p_1p_2}\{(\hbar_{\beta_1}\circ\hbar_{\beta_2})(p_1)\vee\hbar_{\beta_2}(p_2)\})\wedge\gamma_2)\vee\delta_2\\
&\leq (\{(\hbar_{\beta_1}\circ\hbar_{\beta_2})(g_1(l_2(l_1g_1)))\vee\hbar_{\beta_2}(g_1)\}\wedge\gamma_2)\vee\delta_2\\
&= \inf_{g_1(l_2(l_1g_1))=uv}\{\hbar_{\beta_1}(u)\vee\hbar_{\beta_2}(v)\}\vee\hbar_{\beta_2}(g_1)\vee\delta_2\\
&\leq (\{(\hbar_{\beta_1}(g_1)\vee\delta_2)\vee(\hbar_{\beta_2}(g_1)\vee\delta_2)\}\hbar_{\beta_2}(g_1)\vee\delta_2)\wedge\gamma_2\\
&= (((\hbar_{\beta_1}(g_1)\vee\hbar_{\beta_2}(g_1))\wedge\gamma_2)\vee\delta_2\\
&= \hbar_{\beta_1^*}\vee\hbar_{\beta_2^*}(g_1).
\end{aligned}$$

Thus,
$((\beta_1\circ^*\beta_2)\circ^*\beta_1)(g_1) =$
$\left\langle\widetilde{\Im}_{(\beta_1\circ^*\beta_2)\circ^*\beta_1}(g_1)\succeq(\widetilde{\Im}_{\beta_1^*}\widetilde{\wedge}\widetilde{\Im}_{\beta_2^*})(g_1),\hbar_{(\beta_1\circ^*\beta_2)\circ^*\beta_1}(g_1)\leq\left(\hbar_{\beta_1^*}\vee\hbar_{\beta_2^*}\right)(g_1)\right\rangle\supseteq\beta_1\wedge^*\beta_2.$

Now, for the reverse inclusion consider

$$\begin{aligned}
\widetilde{\Im}_{(\beta_1\circ^*\beta_2)\circ^*\beta_1}(g_1) &= (((\widetilde{\Im}_{\beta_1}\widetilde{\circ}\widetilde{\Im}_{\beta_2})\widetilde{\circ}\widetilde{\Im}_{\beta_1})(g_1)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&= ((\operatorname*{rsup}_{g_1=p_1p_2}\{(\widetilde{\Im}_{\beta_1}\widetilde{\circ}\widetilde{\Im}_{\beta_2})(p_1)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(p_2)\})\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&= ((\operatorname*{rsup}_{g_1=p_1p_2}\{\operatorname*{rsup}_{p_1=lm}\{\widetilde{\Im}_{\beta_1}(l)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(m)\}\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(p_2)\})\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&\preceq \operatorname*{rsup}_{g_1=p_1p_2}\{\operatorname*{rsup}_{p_1=lm}\widetilde{\Im}_{\beta_1}(lm)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(lm)\widetilde{\vee}\widetilde{\gamma_1}\}\widetilde{\wedge}(\widetilde{\Im}_{\beta_2}(p_1p_2)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&= \operatorname*{rsup}_{g_1=p_1p_2}\{\widetilde{\Im}_{\beta_1}(p_1)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(p_2)\widetilde{\vee}\widetilde{\gamma_1}\}\widetilde{\wedge}(\widetilde{\Im}_{\beta_2}(p_1p_2)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&\preceq \operatorname*{rsup}_{g_1=p_1p_2}\{\widetilde{\Im}_{\beta_1}(p_1p_2)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(p_1p_2)\widetilde{\vee}\widetilde{\gamma_1}\}\widetilde{\wedge}(\widetilde{\Im}_{\beta_2}(p_1p_2)\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&= (\widetilde{\Im}_{\beta_1}(g_1)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(g_1)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(g_1))\widetilde{\vee}\widetilde{\gamma_1}\widetilde{\wedge}\widetilde{\delta_1}\\
&= (((\widetilde{\Im}_{\beta_1}(g_1)\widetilde{\wedge}\widetilde{\Im}_{\beta_2}(g_1))\widetilde{\vee}\widetilde{\gamma_1})\widetilde{\wedge}\widetilde{\delta_1}\\
&= \widetilde{\Im}_{\beta_1^*}\widetilde{\wedge}\widetilde{\Im}_{\beta_2^*}(g_1),
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\hbar_{(\beta_1\circ^{*}\beta_2)\circ^{*}\beta_1}(g_1) &= (((\hbar_{\beta_1}\circ\hbar_{\beta_2})\circ\hbar_{\beta_1})(g_1)\wedge\gamma_2)\vee\delta_2\\
&= ((\inf_{g_1=p_1p_2}\{(\hbar_{\beta_1}\circ\hbar_{\beta_2})(p_1)\vee\hbar_{\beta_2}(p_2)\})\wedge\gamma_2)\vee\delta_2\\
&= ((\inf_{g_1=p_1p_2}\{\inf_{p_1=lm}\{\hbar_{\beta_1}(l)\vee\hbar_{\beta_2}(m)\}\vee\hbar_{\beta_2}(p_2)\})\wedge\gamma_2)\vee\delta_2\\
&\geq \inf_{g_1=p_1p_2}\{\inf_{g_1=lm}\hbar_{\beta_1}(lm)\vee\hbar_{\beta_2}(lm)\wedge\gamma_2\}\vee(\hbar_{\beta_2}(p_1p_2)\wedge\gamma_2)\vee\delta_2\\
&= \inf_{g_1=p_1p_2}\{\hbar_{\beta_1}(p_1)\vee\hbar_{\beta_2}(p_2)\wedge\gamma_2\}\vee(\hbar_{\beta_2}(p_1p_2)\wedge\gamma_2)\vee\delta_2\\
&\geq \inf_{g_1=p_1p_2}\{\hbar_{\beta_1}(p_1p_2)\vee\hbar_{\beta_2}(p_1p_2)\wedge\gamma_2\}\vee(\hbar_{\beta_2}(p_1p_2)\wedge\gamma_2)\vee\delta_2\\
&= (\hbar_{\beta_1}(g_1)\vee\hbar_{\beta_2}(g_1)\vee\hbar_{\beta_2}(g_1))\wedge\gamma_2\vee\delta_2\\
&= (((\hbar_{\beta_1}(g_1)\vee\hbar_{\beta_2}(g_1))\wedge\gamma_2)\vee\delta_2\\
&= \hbar_{\beta_1^{*}}\vee\hbar_{\beta_2^{*}}(g_1).
\end{aligned}$$

Thus,
$((\beta_1\circ^{*}\beta_2)\circ^{*}\beta_1)(g_1)=$
$\left\langle\widetilde{\Im}_{(\beta_1\circ^{*}\beta_2)\circ^{*}\beta_1}(g_1)\preceq(\widetilde{\Im}_{\beta_1^{*}}\widetilde{\wedge}\widetilde{\Im}_{\beta_2^{*}})(g_1),\hbar_{(\beta_1\circ^{*}\beta_2)\circ^{*}\beta_1}(g_1)\geq\left(\hbar_{\beta_1^{*}}\vee\hbar_{\beta_2^{*}}\right)(g_1)\right\rangle\supseteq\beta_1\wedge^{*}\beta_2.$

Hence, $\beta_1\wedge^{*}\beta_2=(\beta_1\circ^{*}\beta_2)\circ^{*}\beta_1$ for any $(\in_\Gamma,\in_\Gamma\vee q\boldsymbol{\Delta})$-cubic quasi ideals $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ and $\beta_2=\left\langle\widetilde{\Im}_{\beta_2},\hbar_{\beta_2}\right\rangle$ of S.

(iii)⇒(ii) is obvious.

(ii)⇒(i)
Let R_4 and R_3 be the left and quasi ideal of S with left identity. Then $\varkappa_\Gamma^{*\Delta}R_4$ and $\varkappa_\Gamma^{*\Delta}R_3$ are $(\in_\Gamma,\in_\Gamma\vee q\boldsymbol{\Delta})$-cubic left and $(\in_\Gamma,\in_\Gamma\vee q\boldsymbol{\Delta})$-cubic ideal of S. Then, by hypothesis,

$$\varkappa_\Gamma^{*\Delta}R_4\cap R_3=\varkappa_\Gamma^{\Delta}R_4\wedge^{*}\varkappa_\Gamma^{\Delta}R_3=(\varkappa_\Gamma^{\Delta}R_4\circ^{*}\varkappa_\Gamma^{\Delta}R_3)\circ^{*}\varkappa_\Gamma^{\Delta}R_4=\varkappa_\Gamma^{\Delta}(R_4R_3)R_4.$$

Thus, $(R_4R_3)R_4=R_4\cap R_3$. Hence, S is intra-regular by Theorem 2.4. □

Theorem 4.8. *Let $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ be a cubic set of S with left identity, the accompanying conditions are equal,*

(i) S is intra-regular.

(ii) $\beta_1\wedge^{*}\beta_2\subseteq\beta_1\circ^{*}\beta_2$ for all $(\in_\Gamma,\in_\Gamma\vee q\boldsymbol{\Delta})$-cubic bi ideal $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ and $(\in_\Gamma,\in_\Gamma\vee q\boldsymbol{\Delta})$-cubic quasi ideal $\beta_2=\left\langle\widetilde{\Im}_{\beta_2},\hbar_{\beta_2}\right\rangle$ of S.

Proof. (i)⇒(ii)
Let $\beta_1=\left\langle\widetilde{\Im}_{\beta_1},\hbar_{\beta_1}\right\rangle$ and $\beta_2=\left\langle\widetilde{\Im}_{\beta_2},\hbar_{\beta_2}\right\rangle$ be $(\in_\Gamma,\in_\Gamma\vee q\boldsymbol{\Delta})$-cubic bi ideal and $(\in_\Gamma,\in_\Gamma\vee q\boldsymbol{\Delta})$-cubic quasi ideal of an intra-regular $\mathcal{AG}$-groupoid S with left identity. Since S is intra-regular so for each $g_1\in S$ there exist $l_1,l_2\in S$ such that $g_1=(l_1g_1^2)l_2$, and $S=S^2$ so for $l_2\in S$ there exist $s,t\in S$ such that $l_2=st$.

Now, as

$$\begin{aligned} g_1 &= (l_1 g_1^2) l_2 = (l_1(g_1 g_1)) l_2 = (g_1(l_1 g_1)) l_2 = (l_2(l_1 g_1)) g_1 \\ &= [(st)(l_1 g_1)] g_1 = [(g_1 l_1)(ts)] g_1 = [\{(ts) l_1\} g_1] g_1 = [\{(ts) l_1\}((l_1 g_1^2) l_2)] g_1 \\ &= [(l_1 g_1^2)(\{(ts) l_1\} l_2)] g_1 = [\{g_1(l_1 g_1)\}(\{(ts) l_1\} l_2)] g_1 \\ &= [\{(\{(ts) l_1\} l_2)(l_1 g_1)\} g_1] g_1 \\ &= [\{p_1(l_1 g_1)\} g_1] g_1, \text{ where } p_1 = ((ts) l_1) l_2, \end{aligned}$$

and

$$\begin{aligned} p_1(l_1 g_1) &= p_1[l_1\{(l_1 g_1^2) l_2\}] = p_1[(l_1 g_1^2)(l_1 l_2)] = (l_1 g_1^2)[p_1(l_1 l_2)] \\ &= [(l_1 l_2) p_1](g_1^2 l_1) = g_1^2([(l_1 l_2) p_1] l_1) = g_1^2 p_2, \text{ where } p_2 = [(l_1 l_2) p_1] l_1, \end{aligned}$$

therefore, $g_1 = ((g_1^2 p_2) g_1) g_1$, where $p_2 = [(l_1 l_2) p_1] l_1$ and $p_1 = ((ts) l_1) l_2$.

Now, consider

$$\begin{aligned} \widetilde{\Im}_{\beta_1 \circ^{*} \beta_2}(g_1) &= ((\widetilde{\Im}_{\beta_1} \widetilde{\circ} \widetilde{\Im}_{\beta_2})(g_1) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\ &= ((\underset{g_1 = p_1 \circ p_2}{\text{rsup}} \{\widetilde{\Im}_{\beta_1}(p_1) \widetilde{\wedge} \widetilde{\Im}_{\beta_2}(p_2)\}) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\ &\succeq (\{\widetilde{\Im}_{\beta_1}((g_1^2 p_2) g_1) \widetilde{\wedge} \widetilde{\Im}_{\beta_2}(g_1)\} \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\ &\succeq \{(\widetilde{\Im}_{\beta_1}(g_1^2) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} (\widetilde{\Im}_{\beta_2}(g_1) \widetilde{\vee} \widetilde{\gamma_1})\} \widetilde{\wedge} \widetilde{\delta_1} \\ &\succeq (\{(\widetilde{\Im}_{\beta_1}(g_1) \widetilde{\wedge} \widetilde{\delta_1}) \widetilde{\wedge} (\widetilde{\Im}_{\beta_2}(g_1) \widetilde{\wedge} \widetilde{\delta_1})\} \widetilde{\wedge} \widetilde{\delta_1}) \widetilde{\vee} \widetilde{\gamma_1} \\ &= (((\widetilde{\Im}_{\beta_1}(g_1) \widetilde{\wedge} \widetilde{\Im}_{\beta_2}(g_1)) \widetilde{\vee} \widetilde{\gamma_1}) \widetilde{\wedge} \widetilde{\delta_1} \\ &= \widetilde{\Im}_{\beta_1^{*}} \widetilde{\wedge} \widetilde{\Im}_{\beta_2^{*}}(g_1), \end{aligned}$$

and on the other hand,

$$\begin{aligned} \hbar_{\beta_1 \circ^{*} \beta_2}(g_1) &= ((\hbar_{\beta_1} \circ \hbar_{\beta_2})(g_1) \wedge \gamma_2) \vee \delta_2 \\ &= ((\underset{g_1 = p_1 \circ p_2}{\inf} \{\hbar_{\beta_1}(p_1) \vee \hbar_{\beta_2}(p_2)\}) \wedge \gamma_2) \vee \delta_2 \\ &\leq \{(\hbar_{\beta_1}((g_1^2 p_2) g_1) \vee \hbar_{\beta_2}(g_1)) \wedge \gamma_2) \vee \delta_2 \\ &\leq \{(\hbar_{\beta_1}(g_1^2) \wedge \gamma_2) \vee (\hbar_{\beta_2}(g_1) \wedge \gamma_2) \vee \delta_2 \\ &\leq (\hbar_{\beta_1}(g_1) \vee \delta_2) \vee (\hbar_{\beta_2}(g_1) \vee \delta_2) \wedge \gamma_2 \\ &= ((\hbar_{\beta_1}(g_1) \vee \hbar_{\beta_2}(g_1)) \wedge \gamma_2) \vee \delta_2 \\ &= (((\hbar_{\beta_1} \vee \hbar_{\beta_2})(g_1)) \wedge \gamma_2) \vee \delta_2 \\ &= \left(\hbar_{\beta_1^{*}} \vee \hbar_{\beta_2^{*}}\right)(g_1). \end{aligned}$$

Thus,
$\beta_1 \circ^{*} \beta_2(g_1) = \left\langle \widetilde{\Im}_{\beta_1 \circ^{*} \beta_2}(g_1) \succeq \left(\widetilde{\Im}_{\beta_1^{*}} \widetilde{\wedge} \widetilde{\Im}_{\beta_2^{*}}\right)(g_1), \hbar_{\beta_1 \circ^{*} \beta_2}(g_1) \leq \left(\hbar_{\beta_1^{*}} \vee \hbar_{\beta_2^{*}}\right)(g_1) \right\rangle \supseteq \beta_1 \wedge^{*} \beta_2.$

$\beta_1 \wedge^{*} \beta_2 \subseteq \beta_1 \circ^{*} \beta_2$ for all $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic bi ideal $\beta_1 = \left\langle \widetilde{\Im}_{\beta_1}, \hbar_{\beta_1} \right\rangle$ and $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic quasi ideal $\beta_2 = \left\langle \widetilde{\Im}_{\beta_2}, \hbar_{\beta_2} \right\rangle$ of S.

(ii)⇒(i)
Let R_2 and R_3 are the bi and quasi ideals of S with left identity respectively. Then, $\varkappa_\Gamma^{*\Delta} R_2$ and

$\varkappa_{\Gamma}^{*\Delta} R_3$ are $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic bi and $(\in_\Gamma, \in_\Gamma \vee \boldsymbol{q\Delta})$-cubic quasi ideals of S respectively. Then, by hypothesis

$$\varkappa_{\Gamma}^{*\delta} R_2 R_3 = \varkappa_{\Gamma}^{\Delta} R_2 \circ^{*} \varkappa_{\Gamma}^{\Delta} R_3 \leq \varkappa_{\Gamma}^{\Delta} R_2 \wedge^{*} \varkappa_{\Gamma}^{\Delta} R_3 = \varkappa_{\Gamma}^{\Delta} R_2 \cap R_3.$$

Thus, $R_2R_3 \subseteq R_2 \cap R_3$. Hence S is intra-regular by Theorem 4.8. □

5 Conclusion

In this paper we have given some characterizations of the intra-regular $\mathcal{AG}$-groupoids by using the generalized cubic ideals. We will also characterize more classes of $\mathcal{AG}$-groupoids through the given generalized cubic ideals. In future we are aiming to provide more generalizations of such types of ideals.

Conflict of Interest The authors declares there is no conflict of interest regarding the publication of this article.

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